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The Goal

Our objective is to determine the thrust produced by the residual high-pressure air after the ejection of all of the water from the rocket motor. To find the thrust, we need to know the mass flow rate and the exit velocity of the air. To find the mass flow rate, we need to know the density of the exiting air. To find the exit velocity, we need to know whether the flow is subsonic or sonic through the nozzle, which is dependent upon the upstream air pressure. We'll work our way through these in reverse order, starting with the initial conditions for the air in the chamber.

Initial Conditions

Before we can analyze the thrust produced by the air, we need to determine our initial conditions (volume, pressure, and density) at the point in time when the last of the water is expelled. Of these, the volume is a known value (the full chamber volume). The pressure we can find by assuming adiabatic expansion during the water ejection phase (see the basic background section for this). To find the density, we can start by using the ideal gas law (and an assumption that the air starts at near atmospheric temperature) to find the number of moles of gas before the water is ejected, then use

$$\rho = n\hat{M}/V$$

with the known molar mass of air ($\hat{M}$=0.0289645 kg/mol) and the full chamber volume at the water/air transition event. It is also useful to find the air temperature, which can be done by using the ideal gas law and the other conditions.

Choked Flow

As the air passes through the nozzle, the exit velocity will be limited to the local speed of sound. Sonic flow will occur when the pressure within the rocket motor is above what we will call the *critical pressure*, which can be calculated by:

$$P_{cr} = P_{atm}\left(\frac{\gamma+1}{2}\right)^{\left(\frac{\gamma}{\gamma-1}\right)} = 1.893 * P_{atm}$$

If the pressure remaining in the motor chamber is above this value, the flow through the nozzle will be *choked*, the exit velocity will be the local speed of sound, and beyond the nozzle the exhaust gas will become supersonic.

Sonic (Choked) Flow Through the Nozzle

When the flow through the nozzle is choked, the pressure, temperature, and density of the gas at the

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throat will be neither atmospheric nor the same as the motor chamber, but will lie somewhere in between. Making a few simplifications, we can use the following relationships to get a close approximation of the throat conditions (P_e, T_e, ρ_e) based on the upstream chamber conditions (P_a, T_a, ρ_a):

$$P_e(t) = P_a(t)\left(1 + \frac{\gamma - 1}{2} M(t)^2\right)^{\frac{-\gamma}{\gamma-1}}$$

$$T_e(t) = T_a(t)\left(1 + \frac{\gamma - 1}{2} M(t)^2\right)^{-1}$$

$$\rho_e(t) = \rho_a(t)\left(1 + \frac{\gamma - 1}{2} M(t)^2\right)^{\frac{-1}{\gamma-1}}$$

where $M(t)$ is the Mach number, which we assume to be a constant 1.0 for choked flow (see here for more details).

Unchoked (Subsonic) Flow Through the Nozzle

Once the chamber pressure drops below the critical pressure, the throat velocity will drop below the speed of sound. The exit plane pressure will be atmospheric ($P_e = P_{atm}$). To find the exit plane temperature and density, we must first find the Mach number for conditions at the throat since it is no longer sonic. To do this, we can use the same pressure relationship found during choked flow:

$$P_{atm} = P_a(t)\left(1 + \frac{\gamma - 1}{2} M(t)^2\right)^{\frac{-\gamma}{\gamma-1}}$$

but now we need to solve for $M(t)$ given the exit pressure and air pressure. Once we find $M(t)$, we can use that to determine the local temperature and density at the throat of the nozzle:

$$T_e(t) = T_a(t)\left(1 + \frac{\gamma - 1}{2} M(t)^2\right)^{-1}$$

$$\rho_e(t) = \rho_a(t)\left(1 + \frac{\gamma - 1}{2} M(t)^2\right)^{\frac{-1}{\gamma-1}}$$

Speed of Sound

For an ideal gas, we can find the local speed of sound within the gas using the following relationship:

$$c = \sqrt{\frac{\gamma P}{\rho}}$$

where γ is the adiabatic index, P is the pressure, and ρ is the density of the gas. Using the ideal gas law, we could also express this in terms of temperature as

$$c = \sqrt{\frac{\gamma R T}{\hat{M}}}$$

where R is the ideal gas constant and $\hat{M}$ is the molar mass of air (see above).

Exhaust Mass Flow Rate

During either subsonic or choked flow, the air will leave the chamber at a rate of:

$$\frac{\mathrm{d}m}{\mathrm{dt}} = \rho_e(t) A_e v_e(t)$$

Conservation of Linear Momentum

As with the water thrust case, we can use conservation of linear momentum to realize that the thrust effect of the departing mass is the product of the mass flow rate and the exit velocity:

$$F(t) = \frac{\mathrm{d}m}{\mathrm{dt}}\, v_e(t)$$
