Air Expansion

We will begin with the rocket motor, filled with a certain volume of water, \( V_w(0) \), and air, \( V_a(0) \), at an initial pressure, \( P_a(0) \). Upon launch, we will assume adiabatic expansion of the air as water is quickly ejected through the nozzle. The pressure and volume of the air are thus related by:

\[
\frac{P_a(t)}{P_a(0)} = \left( \frac{V_a(t)}{V_a(0)} \right)^{-\gamma}
\]

where \( \gamma = 1.4 \) is the adiabatic index for air.

Water Nozzle Exit Velocity

Recognizing that the water is essentially incompressible, we can use Bernoulli’s equation to find the velocity of the water through the nozzle. Bernoulli’s equation states that at any point in a streamline

\[
\frac{v^2}{2} + \frac{p}{\rho} + gh = \text{constant}
\]

where \( v \) is the velocity, \( p \) is the pressure, \( \rho \) is the density, \( g \) is gravity, and \( h \) is the height above a reference plane. Given that the motor is relatively short, we can ignore gravitational effects. We can therefore apply Bernoulli’s equation to compare the nozzle exit to the air/water interface:

\[
\frac{v_e(t)^2}{2} + \frac{P_{atm}}{\rho} = \frac{v_a(t)^2}{2} + \frac{P_a(t)}{\rho}
\]

where \( v_e \) is the nozzle exit velocity, \( P_{atm} \) is the atmospheric pressure, and \( v_a \) is the velocity of the air/water boundary. Now, solving for \( v_e \):

\[
v_e(t) = \sqrt{v_a(t)^2 + \frac{2(P_a(t) - P_{atm})}{\rho}}
\]

Thrust

The linear momentum of a water particle of mass \( \Delta m \) expelled from the nozzle at velocity of \( v_e(t) \) can be expressed as

\[
\Delta L = \Delta m \, v_e(t)
\]

dividing by \( \Delta t \) and taking the limit as \( t \) goes to zero results in
\[ \dot{L} = \dot{m} \: v_e(t) \]

which is equal to the sum of external forces acting on the water particle, and therefore equal to the thrust acting on the rocket

\[ F(t) = \dot{m} \: v_e(t) \]