1. Introduction

Consider the motion of an axially symmetric rigid top, which has a fixed point (O) at the bottom of the top on its axis of symmetry and the top is acted on by a uniform force such as its weight. Since point O is a fixed point on the top, the top has only three degrees of freedom and we shall use the generalized coordinates $\varphi$, $\psi$ and $\theta$, shown in Figure 8-13 on page 421 of Greenwood’s text, and repeated below.

Note that $\varphi$, which is not marked in this figure, measures rotations about the symmetry axis of the top. Also in this figure, $XYZ$ is a fixed inertial reference frame that has its origin at point O. The principle axes of symmetry for the top
will be used as the body-fixed frame, and this frame has its origin also at point O. In addition, this frame will be denoted by $xyz$, with the $x$ axis along the axis of symmetry of the top.

2. The Moment of Inertia Matrix for the Top

Using the notation described above, we note that the moment of inertia matrix about the point O, which respect to the principle axes of symmetry $xyz$, is diagonal and is given by

$$[I] = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix},$$

with

$$I_{xx} = \int_{\text{body}} ((\hat{x} \cdot \hat{x})r^2 - (\mathbf{r} \cdot \hat{x})(\mathbf{r} \cdot \hat{x}))(\rho(r))dV$$

$$= \int_{\text{body}} (r^2 - x^2)\rho dV = \rho \int_{\text{body}} (y^2 + z^2)dV$$

which we shall denote by $I_a$. Also

$$I_{yy} = \int_{\text{body}} ((\hat{y} \cdot \hat{y})r^2 - (\mathbf{r} \cdot \hat{y})(\mathbf{r} \cdot \hat{y}))(\rho(r))dV$$

$$= \int_{\text{body}} (r^2 - y^2)\rho dV = \rho \int_{\text{body}} (x^2 + z^2)dV$$

which we shall denote by $I_t$, and

$$I_{zz} = \int_{\text{body}} ((\hat{z} \cdot \hat{z})r^2 - (\mathbf{r} \cdot \hat{z})(\mathbf{r} \cdot \hat{z}))(\rho(r))dV$$

$$= \int_{\text{body}} (r^2 - z^2)\rho dV = \rho \int_{\text{body}} (x^2 + y^2)dV$$

which is also equal to $I_t$, because of the assumed symmetry about the $x$ axis, which makes

$$\int_{\text{body}} z^2 dV = \int_{\text{body}} y^2 dV. \quad (1)$$

Therefore we may write the *principle* moment of inertia matrix as

$$[I] = \begin{bmatrix} I_a & 0 & 0 \\ 0 & I_t & 0 \\ 0 & 0 & I_t \end{bmatrix}. \quad (2)$$
3. The Kinetic Energy of the Top

The kinetic energy of the top (as measured from the fixed point $O$) can be computed using

$$T = rac{1}{2} \mathbf{\omega} \cdot \mathbf{I} \cdot \mathbf{\omega}$$  \hspace{1cm} (3a)

which becomes

$$T = \frac{1}{2} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}^T \begin{bmatrix} I_a & 0 & 0 \\ 0 & I_t & 0 \\ 0 & 0 & I_t \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

or simply

$$T = \frac{1}{2} (I_a \omega_x^2 + I_t (\omega_y^2 + \omega_z^2)) \hspace{1cm} (3b)$$

where

$$\mathbf{\omega} = \omega_x \mathbf{x} + \omega_y \mathbf{y} + \omega_z \mathbf{z} \hspace{1cm} (4)$$

is the angular velocity vector of the top as measured by point O and expressed in the body fixed frame.

Now the expression in Equation (4) gives $\mathbf{\omega}$ in terms of the $xyz$ body-fixed frame and we would like to write $\omega_x$, $\omega_y$, and $\omega_z$ in terms of the Euler angles shown in the figure, since we will want to use these Euler angles as the generalized coordinates for this system.

Toward this end, we first note that Figure 7-22 on page 356 and Figure 8-6 on
page 406 of Greenwood’s text are very useful, and these are presented below.

Next we note from the addition theorem for angular velocities, that

\[ \omega = \dot{\psi} + \dot{\theta} + \dot{\phi} \]  

(5a)

and from Figure 8-6, we can see that \( \dot{\psi} \) is along the \( \hat{Z} = \hat{z}' \) axis, \( \dot{\theta} \) is along the
Finally, we also see that rotation of $b$ and $y$ by a simple rotation of $\psi$ about the positive $Z$ axis, which means

$$\dot{\psi} = \psi \dot{Z}, \quad \dot{\theta} = \theta \dot{y}' \quad \text{and} \quad \dot{\varphi} = \varphi \dot{x}.$$  \hspace{1cm} (5b)

But from Figure 8-6, we can see that $\hat{x}'$, $\hat{y}'$, and $\hat{z}'$ are constructed from $\hat{X}$, $\hat{Y}$, and $\hat{Z}$ by a simple rotation of $\psi$ about the positive $Z$ axis, which means

$$\begin{bmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{bmatrix} = \begin{bmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{X} \\ \hat{Y} \\ \hat{Z} \end{bmatrix}.$$  \hspace{1cm} (6a)

In addition, we see that $\hat{x}''$, $\hat{y}''$, and $\hat{z}''$ are constructed from $\hat{x}'$, $\hat{y}'$, and $\hat{z}'$ by a rotation of $\theta$ about the positive $y'$ axis, which means

$$\begin{bmatrix} \hat{x}'' \\ \hat{y}'' \\ \hat{z}'' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \begin{bmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{bmatrix}.$$  \hspace{1cm} (6b)

Finally, we also see that $\hat{x}$, $\hat{y}$, and $\hat{z}$ are constructed from $\hat{x}'$, $\hat{y}'$, and $\hat{z}'$ by a rotation of $\varphi$ about the positive $x''$ axis, which means that

$$\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\varphi) & \sin(\varphi) \\ 0 & -\sin(\varphi) & \cos(\varphi) \end{bmatrix} \begin{bmatrix} \hat{x}'' \\ \hat{y}'' \\ \hat{z}'' \end{bmatrix}.$$  \hspace{1cm} (6c)

Equation (6b) then implies that

$$\begin{bmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}^{-1} \begin{bmatrix} \hat{x}'' \\ \hat{y}'' \\ \hat{z}'' \end{bmatrix} = \begin{bmatrix} \hat{x}'' \\ \hat{y}'' \\ \hat{z}'' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}^{-1} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix}$$

and Equation (6c) gives

$$\begin{bmatrix} \hat{x}'' \\ \hat{y}'' \\ \hat{z}'' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\varphi) & \sin(\varphi) \\ 0 & -\sin(\varphi) & \cos(\varphi) \end{bmatrix}^{-1} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\varphi) & -\sin(\varphi) \\ 0 & \sin(\varphi) & \cos(\varphi) \end{bmatrix}^{-1} \begin{bmatrix} \hat{x}'' \\ \hat{y}'' \\ \hat{z}'' \end{bmatrix}$$

so that

$$\begin{bmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\varphi) & -\sin(\varphi) \\ 0 & \sin(\varphi) & \cos(\varphi) \end{bmatrix}^{-1} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix}.$$
or
\[
\begin{bmatrix}
\dot{x}' \\
\dot{y}' \\
\dot{z}'
\end{bmatrix} =
\begin{bmatrix}
\cos(\theta) & \sin(\theta) \sin(\varphi) & \sin(\theta) \cos(\varphi) \\
0 & \cos(\varphi) & -\sin(\varphi) \\
-\sin(\theta) & \cos(\theta) \sin(\varphi) & \cos(\theta) \cos(\varphi)
\end{bmatrix}
\begin{bmatrix}
x' \\
y' \\
z'
\end{bmatrix}
\] (7)

which means that
\[
\dot{y}' = \cos(\varphi)\dot{y} - \sin(\varphi)\dot{z}
\]

and so
\[
\dot{\theta} = \dot{\theta}\dot{y}' = \dot{\theta}\cos(\varphi)\dot{y} - \dot{\theta}\sin(\varphi)\dot{z}.
\] (8)

We also have from Equation (6a) that
\[
\begin{bmatrix}
\hat{x} \\
\hat{y} \\
\hat{z}
\end{bmatrix} =
\begin{bmatrix}
\cos(\psi) & \sin(\psi) & 0 \\
-\sin(\psi) & \cos(\psi) & 0 \\
0 & 0 & 1
\end{bmatrix}^{-1}
\begin{bmatrix}
\dot{x}' \\
\dot{y}' \\
\dot{z}'
\end{bmatrix} =
\begin{bmatrix}
\cos(\psi) & -\sin(\psi) & 0 \\
\sin(\psi) & \cos(\psi) & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\dot{x}' \\
\dot{y}' \\
\dot{z}'
\end{bmatrix}
\]

and from Equation (7) we have
\[
\begin{bmatrix}
\hat{x} \\
\hat{y} \\
\hat{z}
\end{bmatrix} =
\begin{bmatrix}
\cos(\psi) & -\sin(\psi) & 0 \\
\sin(\psi) & \cos(\psi) & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\cos(\theta) & \sin(\theta) \sin(\varphi) & \sin(\theta) \cos(\varphi) \\
0 & \cos(\varphi) & -\sin(\varphi) \\
-\sin(\theta) & \cos(\theta) \sin(\varphi) & \cos(\theta) \cos(\varphi)
\end{bmatrix}
\begin{bmatrix}
x' \\
y' \\
z'
\end{bmatrix}
\]

which leads to
\[
\hat{z} = -\sin(\theta)\hat{x} + \cos(\theta)\sin(\varphi)\hat{y} + \cos(\theta)\cos(\varphi)\hat{z}
\]

and so
\[
\dot{\psi} = \dot{\psi}\hat{z} = -\dot{\psi}\sin(\theta)\hat{x} + \dot{\psi}\cos(\theta)\sin(\varphi)\hat{y} + \dot{\psi}\cos(\theta)\cos(\varphi)\hat{z}.
\] (9)

Finally we take Equations (5a,b) along with Equations (8) and (9) and write
\[
\omega = \dot{\psi} + \dot{\theta} + \dot{\varphi}
\]
\[
= -\dot{\psi}\sin(\theta)\hat{x} + \dot{\psi}\cos(\theta)\sin(\varphi)\hat{y} + \dot{\psi}\cos(\theta)\cos(\varphi)\hat{z}
\]
\[
+ \dot{\theta}\cos(\varphi)\hat{y} - \dot{\theta}\sin(\varphi)\hat{z} + \dot{\varphi}\hat{x}
\]

or simply
\[
\omega = (\dot{\varphi} - \dot{\psi}\sin(\theta))\hat{x} + (\dot{\theta}\cos(\varphi) + \dot{\psi}\cos(\theta)\sin(\varphi))\hat{y} + (\dot{\psi}\cos(\theta)\cos(\varphi) - \dot{\theta}\sin(\varphi))\hat{z}.
\]
Comparing this with Equation (4), we see that

\[
\begin{align*}
\omega_x &= \dot{\phi} - \dot{\psi} \sin(\theta) \\
\omega_y &= \dot{\theta} \cos(\phi) + \dot{\psi} \cos(\theta) \sin(\phi) \\
\omega_z &= \dot{\psi} \cos(\phi) \cos(\theta) - \dot{\theta} \sin(\phi).
\end{align*}
\] (10)

Putting these into the expression for kinetic energy (Equation (3b)), we now have

\[
T = \frac{1}{2}(I_a \omega_x^2 + I_t (\omega_y^2 + \omega_z^2))
\]

with

\[
\omega_x = \dot{\phi} - \dot{\psi} \sin(\theta) \quad \text{and} \quad \omega_y^2 + \omega_z^2 = \dot{\theta}^2 + \dot{\psi}^2 \cos^2(\theta)
\]

and so

\[
T = \frac{1}{2} I_a (\dot{\phi} - \dot{\psi} \sin(\theta))^2 + \frac{1}{2} I_t (\dot{\theta}^2 + \dot{\psi}^2 \cos^2(\theta)).
\] (11a)

4. The Potential Energy of the Top

The potential energy of the top (see Figure 8.13 on page 421), as measured from point O, is due entirely to its weight and is given by

\[
V = mg l \sin(\theta)
\] (11b)

where \( l \) is the distance between point O and the top’s center of mass C. Note that positive \( \theta \) is assumed to be measured above point O while negative \( \theta \) is measured below point O, and \( \theta = 0 \) is the horizontal plane containing point O.

5. The Lagrangian of the Top and some Constants of the Motion

The Lagrangian of the top is given by \( \mathcal{L} = T - V \), which, via Equations (11a,b) leads to

\[
\mathcal{L} = \frac{1}{2} I_a (\dot{\phi} - \dot{\psi} \sin(\theta))^2 + \frac{1}{2} I_t (\dot{\theta}^2 + \dot{\psi}^2 \cos^2(\theta)) - mg l \sin(\theta).
\] (12)

Lagrange’s equations of motion then yield

\[
\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0, \quad \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0
\]
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\psi}} \right) - \frac{\partial L}{\partial \psi} = 0 \]

but since
\[ \frac{\partial L}{\partial \dot{\psi}} = \frac{\partial L}{\partial \dot{\varphi}} = 0, \]

we see that
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\psi}} \right) = 0 \quad \text{and} \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\varphi}} \right) = 0 \]

which says that
\[ \frac{\partial L}{\partial \dot{\psi}} \quad \text{and} \quad \frac{\partial L}{\partial \dot{\varphi}} \]

are both \textit{constants of the motion}. We shall define the three \textit{generalized momenta}

\[ p_\psi = \frac{\partial L}{\partial \dot{\psi}} = -I_a(\dot{\varphi} - \dot{\psi}\sin(\theta))\sin(\theta) + I_t\dot{\psi}\cos^2(\theta) \quad (13a) \]

and
\[ p_\theta = \frac{\partial L}{\partial \dot{\theta}} = I_t\dot{\theta} \quad (13b) \]

and
\[ p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = I_a(\dot{\varphi} - \dot{\psi}\sin(\theta)) = I_a\omega_x. \quad (13c) \]

and in terms of these, we see that \( p_\psi \) and \( p_\varphi \) are \textit{constants} of the motion. Thus (since \( I_a \) is constant) we see that the spin of the top along its axis of symmetry, which is given by \( \omega_x = \dot{\varphi} - \dot{\psi}\sin(\theta) \) is also a constant of the motion and we shall denote this by \( \Omega \), so that
\[ \omega_x = \dot{\varphi} - \dot{\psi}\sin(\theta) \equiv \Omega. \quad (14) \]

Putting this in Equations (13a and c), we now have
\[ p_\psi = -I_a\Omega\sin(\theta) + I_t\dot{\psi}\cos^2(\theta) \quad \text{and} \quad p_\varphi = I_a\Omega \quad (15) \]

as \textit{constants} of the motion, and solving these for the time derivatives, leads to
\[ \dot{\psi} = \frac{p_\psi + I_a\Omega\sin(\theta)}{I_t\cos^2(\theta)} \quad \text{and} \quad \dot{\varphi} \equiv \Omega + \dot{\psi}\sin(\theta). \quad (16) \]
We also note that the total energy of the system, given by

\[ E = \frac{1}{2} I_a (\dot{\phi} - \dot{\psi} \sin(\theta))^2 + \frac{1}{2} I_t (\dot{\theta}^2 + \dot{\psi}^2 \cos^2(\theta)) + mgl \sin(\theta) \]  

(17)
is also a constant of the motion. Of course since \( \Omega = \dot{\phi} - \dot{\psi} \sin(\theta) \) is also constant, Equation (17) tells us that

\[ E = \frac{1}{2} I_a \Omega^2 + \frac{1}{2} I_t (\dot{\theta}^2 + \dot{\psi}^2 \cos^2(\theta)) + mgl \sin(\theta) \]
or

\[ \frac{1}{2} I_a (\dot{\theta}^2 + \dot{\psi}^2 \cos^2(\theta)) + mgl \sin(\theta) = E - \frac{1}{2} I_a \Omega^2 \equiv E' \]

(18)
is also a constant of the motion. If we replace \( \dot{\psi} \) in this expression by what it equals in Equation (16), we have

\[ \frac{1}{2} I_t \left\{ \ddot{\theta}^2 + \left( \frac{p_\psi + I_a \Omega \sin(\theta)}{I_t \cos^2(\theta)} \right)^2 \cos^2(\theta) \right\} + mgl \sin(\theta) = E' \]

and solving this for \( \dot{\theta}^2 \) gives

\[ \dot{\theta}^2 = \frac{E'}{2I_t} - \left( \frac{p_\psi + I_a \Omega \sin(\theta)}{I_t \cos^2(\theta)} \right)^2 - \frac{2mgl}{I_t} \sin(\theta). \]  

(19)

We also have

\[ \dot{\psi} = \frac{p_\psi + I_a \Omega \sin(\theta)}{I_t \cos^2(\theta)} \]

and

\[ \dot{\phi} = \Omega + \dot{\psi} \sin(\theta) = \Omega + \left( \frac{p_\psi + I_a \Omega \sin(\theta)}{I_t \cos^2(\theta)} \right) \sin(\theta) \]

(20)

so that now each of \( \dot{\psi}, \dot{\phi}, \) and \( \dot{\theta} \) is in terms of the single angle \( \theta \) and various constants of the motion such as: \( I_a, I_t, m, g, l, \Omega, p_\psi, \) and \( E' \). Note also that the Lagrange equation

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0 \]
yields

\[ I_t \ddot{\theta} + I_a \dot{\psi} \cos(\theta)(\dot{\phi} - \dot{\psi} \sin(\theta)) + I_t \dot{\psi}^2 \sin(\theta) \cos(\theta) + mgl \cos(\theta) = 0 \]
or, since $\Omega = \dot{\phi} - \dot{\psi} \sin(\theta)$, we have

$$I_t \ddot{\theta} + I_a \dot{\psi} \Omega \cos(\theta) + I_t \dot{\psi}^2 \sin(\theta) \cos(\theta) + mgl \cos(\theta) = 0$$

which is recorded for completeness, but is not really needed because of Equation (19).

6. An Analysis of the Motion of the Top

A summary of the top’s equations of motion are now presented and these are:

$$\Omega = \dot{\phi} - \dot{\psi} \sin(\theta), \quad p_\phi = I_a \Omega$$

(21a)

and $p_\psi$ are constants in time, and

$$\dot{\psi} = \frac{p_\psi + p_\phi \sin(\theta)}{I_t \cos^2(\theta)}$$

(21b)

and

$$\dot{\phi} = \frac{p_\phi}{I_a} + \frac{p_\psi + p_\phi \sin(\theta)}{I_t \cos^2(\theta)} \sin(\theta)$$

(21c)

and

$$\dot{\theta}^2 = \frac{E'}{2I_t} - \left( \frac{p_\phi + p_\psi \sin(\theta)}{I_t \cos(\theta)} \right)^2 - \frac{2mgl}{I_t} \sin(\theta)$$

(21d)

give the rate in which $\psi$, $\phi$, and $\theta$ are changing in time. We now find it convenient to introduce the constants

$$a = \frac{p_\psi}{I_t}, \quad b = \frac{p_\phi}{I_t}, \quad c = \frac{2mgl}{I_t}, \quad e = \frac{E'}{2I_t},$$

(22)

which makes the Equations (21b,c,d) reduce to

$$\dot{\psi} = \frac{a + b \sin(\theta)}{\cos^2(\theta)}$$

(23a)

and

$$\dot{\phi} = \frac{(a + b \sin(\theta)) \sin(\theta)}{\cos^2(\theta)} + \frac{I_a}{I_t} a$$

(23b)

and

$$\dot{\theta}^2 = e - c \sin(\theta) - \left( \frac{a + b \sin(\theta)}{\cos(\theta)} \right)^2.$$  

(23c)
Note that $a$ and $b$ have dimensions of inverse time (1/seconds), while $c$ and $e$ have dimensions of inverse time squared (1/seconds$^2$). Let us further make the substitution

$$u = \sin(\theta) \quad \text{with} \quad \dot{u} = \dot{\theta} \cos(\theta) = \dot{\theta} \sqrt{1 - u^2},$$

(24)

and then Equations (23a,b,c) become

$$\dot{\psi} = \frac{a + bu}{1 - u^2}$$

(25a)

and

$$\dot{\phi} = \frac{(a + bu)u}{1 - u^2} + \frac{I_t a}{I_a},$$

(25b)

and

$$\frac{\dot{u}^2}{1 - u^2} = e - \frac{(a + bu)^2}{1 - u^2} - cu$$

or

$$\dot{u}^2 = f(u) \equiv (e - cu)(1 - u^2) - (a + bu)^2.$$  

(25c)

6.1. Properties of the Polynomial $f(u)$

By the definition of $u$ in Equation (24), we see that $-1 \leq u \leq +1$, and by the definition of $f(u)$ in Equation (25c), we see that $f$ is a cubic polynomial in $u$. You should also notice that

$$f(u) \equiv cu^3 - (e + b^2)u^2 - (c + 2ab)u + (e - a^2)$$

with

$$c = \frac{2mgl}{I_t} > 0.$$

Treat $f$ as a mathematical polynomial in $u$, we see that for very large mathematical values of $|u|$, 

$$f(u) \simeq cu^3$$

and so we find that $f(u) < 0$ for $u < 0$ and $|u|$ large, and $f(u) > 0$ for $u > 0$ and $|u|$ large. We also note from Equation (25c) that

$$f(-1) = -(a - b)^2 < 0 \quad \text{and} \quad f(+1) = -(a + b)^2 < 0.$$
The fact that \( f(+1) \) is negative and \( f(u) \) is positive for large positive values of \( u \), allows us to conclude that \( f \) must have at least one real root at a value of \( u \) larger than 1, and let us call this \( u_3 \), so that \( f(u_3) = 0 \) and \( u_3 > 1 \). We also know that since \( u = \sin(\theta) \), any real physical solutions corresponding to the top’s motion must have \(-1 \leq u \leq +1\) and any real physical solutions must have \( \dot{u}^2 = f(u) \geq 0 \).

Thus we must have \( f(u) \) positive somewhere in the region \(-1 \leq u \leq +1\), and so \( f \) must have another real root between \( u = -1 \) and \( u = +1 \) since \( f(-1) < 0 \) and \( f(+1) < 0 \). This, along with the fact that \( f \), which is a cubic polynomial, has exactly three roots and, since all coefficients of \( f \) are real numbers, \( f \) cannot have only one complex root, and all of this tells us that the other two roots of \( f \) (besides \( u_3 \)) must also be real and must lie in the region \(-1 \leq u \leq +1\). We shall call these \( u_1 \) and \( u_2 \) with \( u_1 \leq u_2 \). Thus we may write \( f \) as

\[
\dot{u}^2 = f(u) = c(u - u_1)(u - u_2)(u - u_3)
\]  

(26a)

with \( c > 0 \), and

\[
-1 \leq u_1 \leq u_2 \leq +1 \leq u_3.
\]  

(26b)

A typical plot of \( f \) versus \( u \) for \( u_1 \neq u_2 \), is as shown below.

A Typical Plot of \( f \) versus \( u \) for \( u_1 \neq u_2 \)
The regions \(-1 \leq u \leq +1\) is between the two dotted vertical lines

Note that in this case, \( \dot{u}^2 = f(u) \geq 0 \) in the physical region \((-1 \leq u \leq +1\)) only
for $u_1 \leq u \leq u_2$. A typical plot of $f$ versus $u$ for $u_1 = u_2 < 0$, is as shown below.

Note that in this case, $\dot{u}^2 = f(u) \geq 0$ in the physical region ($-1 \leq u \leq +1$) only for $u = u_1 = u_2$. A typical plot of $f$ versus $u$ for $u_1 = u_2 > 0$, is as shown below.

Note that in this case, $\dot{u}^2 = f(u) \geq 0$ in the physical region ($-1 \leq u \leq +1$) only for $u = u_1 = u_2$. 
6.2. The Path of a Point on the Symmetry Axis

Let us now consider the path of a point P located on the axis of symmetry at a unit distance from the fixed point O. From Figure 8-13, we then see that \( u = \sin(\theta) \) is simply the height of the point P above the horizontal plane containing point O when \( u > 0 \), and below the horizontal plane containing point O when \( u < 0 \). The values of \( u_1 \) and \( u_2 \) are points for which \( f(u) = 0 \), and hence \( \dot{u} = 0 \), and these are called turning points in the motion, and the entire motion is confined between \( u = u_{\min} = u_1 \) and \( u = u_{\max} = u_2 \), and a typical phase-plane plot of \( \dot{u} \) versus \( u \) is given by

\[
\dot{u} = \pm \sqrt{f(u)}
\]  

(27a)

for \( u_1 \leq u \leq u_2 \), and is shown below as a \( u-\dot{u} \) phase-plane plot.

![Typical Phase-Plane Plot](image)

The fact that this curve is closed indicates that the motion of P must be periodic with a period computed as

\[
\tau = \int_{\text{one cycle}} \frac{1}{\pm \sqrt{f(u)}} du.
\]

or

\[
\tau = \int_{u_2}^{u_1} \frac{-1}{\sqrt{f(u)}} du + \int_{u_1}^{u_2} \frac{1}{\sqrt{f(u)}} du.
\]
Using the symmetry in the phase-plane curve, we may write this as

$$\tau = 2 \int_{u_1}^{u_2} \frac{du}{\sqrt{f(u)}} = 2 \int_{u_1}^{u_2} \frac{du}{\sqrt{c(u-u_1)(u-u_2)(u-u_3)}}$$

or, after putting in the expression for $c$ (Equation (22)), we have

$$\tau = \sqrt{\frac{2I_t}{mgl}} \int_{u_1}^{u_2} \frac{du}{\sqrt{(u-u_1)(u-u_2)(u-u_3)}}$$

which could be expressed in terms of elliptic integrals.

6.3. An Analogy With Mass-Spring Motion

The horizontal motion of a mass on a spring with no friction is governed by the energy equation

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

which, when written as

$$\frac{\dot{x}^2}{2E/m} + \frac{x^2}{2E/k} = 1$$

shows it to be an ellipse in the $x$-$\dot{x}$ phase plane, and is shown in the figure below.

Phase-Plane Plot of $\dot{x}$ (vertical) versus $x$ (horizontal) for the horizontal motion of a mass on a spring. The motion proceeds in the clockwise direction.
This plot also show turning points when $\dot{x} = 0$ at

$$x_1 = x_{\min} = -\sqrt{\frac{2E}{k}} \equiv -x_0 \quad \text{and} \quad x_2 = x_{\max} = +\sqrt{\frac{2E}{k}} \equiv +x_0.$$  

In addition, the period of the motion can be computed by taking the equation

$$\frac{\dot{x}^2}{2E/m} + \frac{x^2}{2E/k} = 1$$

and writing it as

$$\dot{x} = \pm\sqrt{\frac{2E - kx^2}{m}} = \frac{dx}{dt}$$

resulting in

$$dt = \pm\sqrt{\frac{m}{2E - kx^2}} dx$$

or

$$\tau = \int_{\text{one cycle}} \pm\sqrt{\frac{m}{2E - kx^2}} dx$$

or

$$\tau = \sqrt{\frac{m}{k}} \int_{\text{one cycle}} \frac{\pm dx}{\sqrt{x_0^2 - x^2}} = 2\sqrt{\frac{m}{k}} \int_{-x_0}^{x_0} \frac{dx}{\sqrt{x_0^2 - x^2}}$$

which, in this case does not require elliptic integrals, and reduces to just

$$\tau = 2\pi \sqrt{\frac{m}{k}}$$

and is the expected result.

6.4. Back To The Top Problem

If we represent the path of P on the surface of the unit sphere, then we find that P will remain between two horizontal latitude circles given by $\theta = \theta_1$ and $\theta = \theta_2$ corresponding to the extreme values of $u_1 = \sin(\theta_1)$ and $u_2 = \sin(\theta_2)$. See,
The path of P can be classified as one of three general types. Before describing these, we first define

\[ u_0 \equiv -\frac{a}{b} = -\frac{p_\psi}{p_\phi}, \tag{28} \]

and using this, we may rewrite Equation (25a,b,c) as

\[ \dot{\psi} = \frac{b(u - u_0)}{1 - u^2} \]  \hspace{1cm} (29a)

and

\[ \dot{\phi} = \frac{bu(u - u_0)}{1 - u^2} + \frac{I_\phi}{I_\alpha}, \tag{29b} \]

and

\[ \dot{u}^2 = f(u) \equiv (e - cu)(1 - u^2) - b^2(u - u_0)^2. \tag{29c} \]

Equation (29a) shows that \( \dot{\psi} = 0 \) when \( u = u_0 \neq \pm1 \). Hence the motion of the symmetry axis when \( u = u_0 \) (i.e., \( \theta = \theta_0 \)) is such that the path of P is tangent to a vertical unit circle of "longitude" (see Figure 8-13). Note that in the extreme case when \( u_0 = 1 \), we have

\[ \lim_{u \to 1^-} \dot{\psi} = \lim_{u \to 1^-} \frac{b(u - 1)}{1 - u^2} = -\lim_{u \to 1^-} \frac{b}{1 + u} = -\frac{1}{2} b \neq 0. \]

If \( u_2 < u_0 \), then \( u_0 \) lies outside the physical range of \( u \) values (which recall is \( u_1 \leq u \leq u_2 \)), and therefore the precessional rate \( \dot{\psi} \) does not equal zero at any time during the motion, and this is shown in Figure 8-15(a) in Greenwood’s text which shows the path of P during the motion for positive \( \Omega \).
If $u_1 < u_0 < u_2$, then $\dot{\psi}$ is zero at $u = u_0$, which means that $\dot{\psi}$ is zero twice during each nutation cycle at $\theta_0 = \theta_0^\prime$ and at $\theta_0^\prime = \pi - \theta_0$, where $\sin(\theta_0) = \sin(\theta_0^\prime) = u_0$. Such a path is shown in Figure 8-15(b) in Greenwood’s text.

Next we consider the case when $u_0 = u_2$ (which says that $\theta_0 = \theta_2$), which is referred to as cuspidal motion, and is illustrated in Figure 8-15(c) in Greenwood’s text.

7. Cuspidal Motion of the Top: Calculating The Turning Points

For this motion we note that both $\dot{u}^2 = f(u_2) = 0$ and $\dot{\psi} = 0$ when $u = u_0 = u_2$ (i.e., $\theta = \theta_0 = \theta_2$ and when $\theta_0^\prime = \pi - \theta_2$). Since $\dot{u} = \dot{\theta} \cos(\theta)$, we see that when $u = u_0$ (i.e., $\dot{u} = 0$), we have either $\dot{\theta} = 0$ or $\cos(\theta) = 0$. Let us now consider each of these two cases.

7.1. Case #1: Cuspidal Motion of the Top with $u = u_0$ when $\dot{\theta} = 0$

The first case corresponds to $u = u_0$ when $\dot{\theta} = 0$. Using the definition of $u_0$ in Equation (29c), we see that

$$f(u) = (e - cu)(1 - u^2) - b^2(u_0 - u)^2.$$  

But we also have

$$\dot{u}^2 = f(u_0) = (e - cu_0)(1 - u_0^2) = 0$$

which leads to $e = cu_0$ and says that

$$u_0 = \frac{e}{c} = \frac{E'}{2I_t} \times \frac{I_t}{2mgl} = \frac{E'}{4mgl} = \frac{p_\psi}{p_\phi}$$

so that

$$E' = E - \frac{1}{2} I_a \Omega^2 = -\frac{4mglp_\psi}{p_\phi}$$

or, since $p_\phi = I_a \Omega$, 

$$E = \frac{1}{2} I_a \Omega^2 - \frac{4mglp_\psi}{I_a \Omega}.$$ 

Replacing $e$ by $cu_0$ in Equation (30), we have

$$f(u) = c(u_0 - u)(1 - u^2) - b^2(u_0 - u)^2$$
or
\[ f(u) = (u_0 - u)(c(1 - u^2) - b^2(u_0 - u)) \]
and setting this equal to zero (in order to get the turning points) yields, besides \( u = u_0 \), the equation
\[ c(1 - u^2) - b^2(u_0 - u) = 0 \]
or
\[ u^2 - \frac{b^2}{c} u + \frac{b^2}{c} u_0 - 1 = 0 \]
or
\[ u^2 - 2\lambda u + 2\lambda u_0 - 1 = 0 \quad \text{with} \quad \lambda = \frac{b^2}{2c} \]
But recall that
\[ b = \frac{p_x}{I_t} \quad \text{and} \quad c = \frac{2mgl}{I_t} \]
and so
\[ \lambda = \frac{b^2}{2c} = \left( \frac{p_x}{I_t} \right)^2 \frac{I_t}{4mgl} = \frac{p_x^2}{4I_t mgl} = \frac{(I_0 \Omega)^2}{4I_t mgl} > 0. \]
Solving the equation \( u^2 - 2\lambda u + 2\lambda u_0 - 1 = 0 \) gives
\[ u = \frac{2\lambda \pm \sqrt{(2\lambda)^2 - 4(2\lambda u_0 - 1)}}{2} \]
or
\[ u = \lambda \pm \sqrt{\lambda^2 - 2\lambda u_0 + 1} \]
This leads to
\[ u_1 = \lambda - \sqrt{\lambda^2 - 2\lambda u_0 + 1}, \quad u_2 = u_0, \quad u_3 = \lambda + \sqrt{\lambda^2 - 2\lambda u_0 + 1} \]
as the three roots of \( f(u) \). Of course for \(-1 < u_0 \leq 1\), we find that
\[-1 \leq u_1 \leq u_2 \leq 1 \leq u_3 \]
regardless of the value of \( \lambda \). The fact that \( u_3 \geq 1 \) can be checked by writing \( u_3 \) as
\[ u_3 = 1 + (\lambda - 1) + \sqrt{\lambda^2 - 2\lambda u_0 + 1} - 2\lambda + 2\lambda \]
or
\[ u_3 = 1 + (\lambda - 1) + \sqrt{(\lambda - 1)^2 + 2\lambda(1 - u_0)} \]
showing that
\[ u_3 = 1 + (\lambda - 1) + \sqrt{(\lambda - 1)^2 + \text{positive}} \geq 1 \]
since \( u_0 \leq 1 \) (i.e., \( 1 - u_0 \geq 0 \)) and
\[ (\lambda - 1) + \sqrt{(\lambda - 1)^2 + \text{positive}} \geq 0 \]
for any choice of \( \lambda - 1 \). Thus we have
\[ u_0 = -\frac{p\dot{\psi}}{p_\phi} = \frac{E'}{4mgl} = \frac{2E - I_\alpha \Omega^2}{8mgl} > 0 \quad , \quad \lambda = \frac{(I_\alpha \Omega)^2}{4I_t mgl} > 0 \]
and
\[
\begin{align*}
u_1 &= \lambda - \sqrt{\lambda^2 - 2\lambda u_0 + 1} \\
u_2 &= u_0 > 0, \\
u_3 &= \lambda + \sqrt{\lambda^2 - 2\lambda u_0 + 1} > 1
\end{align*}
\]
as the places where \( \dot{u}^2 = f(u) = 0 \). Of course the point at \( u = u_3 \) is non-physical since it is larger than 1. Note also that the turning point at
\[ u_1 = \lambda - \sqrt{\lambda^2 - 2\lambda u_0 + 1} \]
could be positive or negative depending on the values of \( \lambda \) and \( u_0 \). In fact, we see that
\[ u_1 = \lambda - \sqrt{\lambda^2 - 2\lambda u_0 + 1} = \lambda - \sqrt{\lambda^2 + (1 - 2\lambda u_0)} < 0 \]
when \( 1 - 2\lambda u_0 > 0 \), or
\[ 2\lambda u_0 = \frac{2(I_\alpha \Omega)^2}{4I_t mgl} \frac{2E - I_\alpha \Omega^2}{8mgl} < 1 \]
which reduces to
\[ E < \frac{1}{2} I_\alpha \Omega^2 + \frac{1}{2} I_t \left( \frac{4mgl}{I_\alpha \Omega} \right)^2. \]
A typical \( u-\dot{u} \) phase plane plot having \( u_1 < 0 \) looks ”egg-shaped” and is shown in
the figure below.

Consequently, we find that \( u_1 > 0 \) when

\[
E > \frac{1}{2} I_a \Omega^2 + \frac{1}{2} I_t \left( \frac{4mgl}{I_a \Omega} \right)^2
\]

and a typical \( u - \dot{u} \) phase-plane plot of this case is shown in the following figure.
7.2. Case #2: Cuspidal Motion of the Top with $u = u_0$ when $\cos(\theta) = 0$

This motion has $\dot{\theta} \neq 0$ and $u = u_0 = \pm 1$, since $u = \sin(\theta)$ and $\cos(\theta) = 0$ gives

$$u = \sin(\theta) = \pm \sqrt{1 - \cos^2(\theta)} = \pm 1.$$ 

7.2a. Case #2a: Cuspidal Motion of the Top with $u_0 = +1$ and $\theta_0 = \pi/2$

The case of $u_0 = +1$ has $\theta_0 = \pi/2$ and this applies when the symmetry axis passes through the upper vertical position with non-zero angular velocity so that the center of mass of the top is above point O. Here we have $a = -b$, and since, of course (see Equation (23c))

$$\dot{\theta}^2 = e - c \sin(\theta) - \left( \frac{a + b \sin(\theta)}{\cos(\theta)} \right)^2 > 0$$

and $a + b \sin(\theta_0) = a + b = 0$ at this moment, we have

$$\dot{\theta}^2 = e - c \sin(\theta_0) = e - c > 0$$

which means that $e > c$. Then we may express $f(u)$ as

$$f(u) = (e - cu)(1 - u^2) - (a + bu)^2$$
$$= (e - cu)(1 - u^2) - b^2(1 - u)^2$$
$$= (1 - u)((e - cu)(1 + u) - b^2(1 - u))$$

which leads to the turning point at $u = u_2 = +1$, and the turning points at $u$ determined from the quadratic equation

$$(e - cu)(1 + u) - b^2(1 - u) = 0$$

which leads to

$$u^2 - \left( \frac{b^2 + e}{c} - 1 \right) u + \left( \frac{b^2 - e}{c} \right) = 0$$

or

$$u_1 = \frac{1}{2} \left( \frac{b^2 + e}{c} - 1 \right) - \frac{1}{2} \sqrt{\left( \frac{b^2 + e}{c} - 1 \right)^2 - 4 \left( \frac{b^2 - e}{c} \right)} < 1 = u_2$$
and

\[ u_3 = \frac{1}{2} \left( \frac{b^2 + e}{c} - 1 \right) + \frac{1}{2} \sqrt{\left( \frac{b^2 + e}{c} - 1 \right)^2 - 4 \left( \frac{b^2 - e}{c} \right)} > 1, \]

which has no physical meaning. Note that a little algebra checks that \( u_3 > 1 \) follows from \( e > c \). Typical \( u-\dot{u} \) phase plane plots look "egg-shaped" and are shown in the next two figures.

![Typical Phase-Plane Plot of \( \dot{u} \) versus \( u \). The motion proceeds in the clockwise direction and this has \( u_1 < 0 \) and \( u_2 = u_0 = 1 \).](image1)

and

![Typical Phase-Plane Plot of \( \dot{u} \) versus \( u \). The motion proceeds in the clockwise direction and this has \( 0 < u_1 < u_2 = u_0 = 1 \).](image2)
7.2b. Case #2b: Cuspidal Motion of the Top with $u_0 = -1$ and $\theta_0 = -\pi/2$

The case of $u_0 = -1 = \sin(\theta_0)$ has $\theta_0 = -\pi/2$ and this applies when the symmetry axis passes through the bottom vertical position with non-zero angular velocity so that the center of mass of the top is now below point O. Here we have $a = b$, and since (see Equation (23c))

$$\dot{\theta}^2 = e - c\sin(\theta) - \left(\frac{a + b\sin(\theta)}{\cos(\theta)}\right)^2 > 0$$

and $a + b\sin(\theta_0) = a - b = 0$ at this moment, we have

$$\dot{\theta}^2 = e - c\sin(\theta_0) = e + c > 0.$$  

We may express $f(u)$ as

$$f(u) = (e - cu)(1 - u^2) - (a + bu)^2$$

$$= (e - cu)(1 - u^2) - b^2(1 + u)^2$$

$$= (1 + u)((e - cu)(1 - u) - b^2(1 + u))$$

which leads to the turning point at $u = u_1 = -1$, and to the turning points at $u$, where

$$(e - cu)(1 - u) - b^2(1 + u) = 0,$$

which reduces

$$u^2 - \left(\frac{b^2 + e}{c} + 1\right)u - \left(\frac{b^2 - e}{c}\right) = 0$$

and results in

$$u_2 = \frac{1}{2}\left(\frac{b^2 + e}{c} + 1\right) - \frac{1}{2}\sqrt{\left(\frac{b^2 + e}{c} + 1\right)^2 + 4\left(\frac{b^2 - e}{c}\right)} > -1 = u_0$$

and

$$u_3 = \frac{1}{2}\left(\frac{b^2 + e}{c} + 1\right) + \frac{1}{2}\sqrt{\left(\frac{b^2 + e}{c} + 1\right)^2 + 4\left(\frac{b^2 - e}{c}\right)} > 1.$$  

Of course the value of $u_3$ has no physical meaning, and a little algebra shows that $u_2 > -1$ follows from the fact that $e + c > 0$, and $u_3 > 1$ follows from the fact
that \( c > 0 \). Typical \( u-\dot{u} \) phase plane plots look "egg-shaped" and are shown in the next two figures.

![Typical Phase-Plane Plot of \( \dot{u} \) versus \( u \). The motion proceeds in the clockwise direction and this has \( u_1 = u_0 = -1 < u_2 < 0 \).](image)

and

![Typical Phase-Plane Plot of \( \dot{u} \) versus \( u \). The motion proceeds in the clockwise direction and this has \( u_1 = u_0 = -1 < 0 < u_2 < 1 \).](image)

### 7.3. Stability of the Top’s Motion Near The Positive Vertical

An example of top motion of particular interest occurs when the axis of symmetry points vertically upward, which has \( \theta_0 = \pi/2 \) and \( \dot{\theta} = 0 \), so that \( u_0 = u_2 = +1 \).
We would like to investigate the stability of this motion near this point. This comes under what we called Case #1 above with \( u_0 = 1 \), and so we have as the roots of \( f(u) \) and hence the turning points of the motion at

\[
\begin{align*}
  u_1 &= \lambda - \sqrt{\lambda^2 - 2\lambda u_0 + 1} = \lambda - \sqrt{\lambda^2 - 2\lambda + 1} = \lambda - |\lambda - 1| = 2\lambda - 1 \\
  u_2 &= u_0 = 1, \text{ and} \\
  u_3 &= \lambda + \sqrt{\lambda^2 - 2\lambda u_0 + 1} = \lambda + \sqrt{\lambda^2 - 2\lambda + 1} = \lambda + |\lambda - 1| = 1
\end{align*}
\]

when \( \lambda \leq 1 \), while

\[
\begin{align*}
  u_1 &= \lambda - \sqrt{\lambda^2 - 2\lambda u_0 + 1} = \lambda - \sqrt{\lambda^2 - 2\lambda + 1} = \lambda - |\lambda - 1| = 1 \\
  u_2 &= u_0 = 1, \text{ and} \\
  u_3 &= \lambda + \sqrt{\lambda^2 - 2\lambda u_0 + 1} = \lambda + \sqrt{\lambda^2 - 2\lambda + 1} = \lambda + |\lambda - 1| = 2\lambda - 1
\end{align*}
\]

as the roots of \( f(u) \) and hence the turning points of the motion when \( \lambda \geq 1 \). Let us consider the two cases having \( \lambda < 1 \) and \( \lambda \geq 1 \) in more detail. Recall that the definition of \( \lambda \) requires that \( 0 < \lambda \).

**7.3a. Case #1:** \( 0 < \lambda < 1 \), or \( (I_a \Omega)^2 < 4I_m gl \)

For this case, we have

\[-1 \leq u_1 = 2\lambda - 1 \leq 1\]

and \( u_2 = u_3 = 1 \). The fact that

\[f(u) = (e - cu)(1 - u^2) - (a + bu)^2\]

has a **double root** at \( u = 1 \) means that \( f(1) = -(a + b)^2 = 0 \) and

\[f'(1) = -2(e - c) - 2b(a + b) = 0.\]

These lead to \( a + b = 0 \) and \( e - c = 0 \), and then

\[f(u) = c(u - 1)^2(u - (2\lambda - 1)) = \dot{u}^2.\]

A typical \( u-\dot{u} \) phase-plane plot of this is shown in the figure below and it shows that a small perturbation off of \( u_2 = 1 \) causes the motion to move toward the other turning point at \( u_1 = 2\lambda - 1 \), and then back to the turning point at \( u_2 = 1 \),
and so on. Therefore if the motion starts at $u_2 = 1$, as small perturbation is enough to have it move off of $u_2 = 1$, and hence the motion is considered \textit{unstable} at $u_2 = 1$, since a small perturbation off of $u = 1$ leads to a trip around the phase-plane curve.

![Typical Phase-Plane Plot of $\dot{u}$ versus $u$. The motion proceeds in the clockwise direction and this has $u_1 = 2\lambda - 1 < 1$, $u_2 = u_3 = 1$.](image)

\textbf{7.3b. Case \#2: $1 \leq \lambda$, or $(I_a\Omega)^2 \geq 4I_t mgl$}

For this case, we still have $a + b = 0$ and $e - c = 0$ since $u = 1$ is still a double root, but now we also have $u_3 = 2\lambda - 1 \geq 1$. Then

$$f(u) = c(u - 1)^2(u - (2\lambda - 1)) = \dot{u}^2.$$  

and since $u - (2\lambda - 1) < 0$ for $-1 \leq u \leq +1$, this motion must be \textit{stable} at $u = 1$, since $u_1 = u_2 = 1$ is the \textit{only physical point} for which

$$\dot{u}^2 = f(u) = c(u - 1)^2(u - (2\lambda - 1)) \geq 0$$

A top which is spinning in this case is called a \textit{sleeping top} since the top appears to be not moving at all.

\textbf{7.4. Instability of the Top’s Motion Near The Negative Vertical}

Another example of the top’s motion of particular interest occurs when the axis of symmetry points vertically downward, which has $\theta_0 = -\pi/2$ and $\dot{\theta} = 0$, so
that \( u_0 = u_1 = -1 \). We would like to investigate the stability of this motion near this point. Starting with

\[
f(u) = (e - cu)(1 - u^2) - b^2(u_0 - u)^2
\]

we have

\[
f(u) = (e - cu)(1 - u^2) - b^2(-1 - u)^2 = (1 + u)((e - cu)(1 - u) - b^2(1 + u))
\]

and so one turning point is at \( u_0 = u_1 = -1 \) and the other turning point occurs when

\[
(e - cu)(1 - u) - b^2(1 + u) = 0
\]

resulting in

\[
u^2 - \left(\frac{b^2 + e}{c} + 1\right)u - \left(\frac{b^2 - e}{c}\right) = 0
\]

leading to

\[
u_2 = \frac{1}{2} \left(\frac{b^2 + e}{c} + 1\right) - \frac{1}{2} \sqrt{\left(\frac{b^2 + e}{c} + 1\right)^2 + 4 \left(\frac{b^2 - e}{c}\right)},
\]

which is always larger than \(-1\) when \( e + c > 0 \), which is true here. Note that \( u_2 < 0 \) when \( b^2 > e \), and \( 0 < u_2 < 1 \) when \( b^2 < e \). Typical phase-plane \( u - \dot{u} \) plots are shown below for the two cases when \(-1 < u_2 < 0\)

![Typical Phase-Plane Plot of \( \dot{u} \) versus \( u \). The motion proceeds in the clockwise direction and this has \( u_1 = -1 \) and \(-1 < u_2 < 0\).](image-url)
and when $0 < u_2 < 1$,

Typical Phase-Plane Plot of $\dot{u}$ versus $u$.
The motion proceeds in the clockwise direction and this has $u_1 = -1$ and $-1 < u_2 < 0$.

and these illustrate that motion around $u = -1$ is never stable, since a small perturbation off of $u = -1$ leads to a trip around the phase-plane curve.

The student is encouraged to look into Greenwood’s text for further discussions involving the motion of an axially symmetric top.