Class Notes for Advanced Dynamics  
(MEAM535)  
Michael A. Carchidi  
December 9, 2009  

Equilibrium Points, Small Perturbations and Linear Stability Analysis  

The following notes were initially based around the text entitled: Theory of Vibrations with Applications (5th Edition) by William T. Thomson and Marie Dillon Dahleh.  

1. Lagrange’s Equations of Motion  

We have seen in lecture that if a system has \( n \) degrees-of-freedom described by the generalized coordinate \( q_1, q_2, q_3, \ldots, q_n \), then the kinetic energy of the system can be expressed as a function of the \( q_k \)'s and \( \dot{q}_k \)'s so that  
\[
T = T(q_1, q_2, q_3, \ldots, q_n, \dot{q}_1, \dot{q}_2, \dot{q}_3, \ldots, \dot{q}_n)
\]  
and if the forces, acting on the system described by the \( n \) generalized coordinates \( q_1, q_2, q_3, \ldots, q_n \), are conservative, which is the case with freely vibrational systems without damping, then we can find a total potential function  
\[
V = V(q_1, q_2, q_3, \ldots, q_n).
\]  
the Lagrangian of the system is then defined as  
\[
\mathcal{L} = T - V = \mathcal{L}(q_1, q_2, q_3, \ldots, q_n, \dot{q}_1, \dot{q}_2, \dot{q}_3, \ldots, \dot{q}_n)
\]  
and the equations of motion become  
\[
\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \right) - \frac{\partial \mathcal{L}}{\partial q_j} = 0
\]
for \( j = 1, 2, 3, \ldots, n \). These \( n \) second-order differential equations are coupled in the \( q_k \)'s.

2. Example: A Cart-Rod System

A cart of mass \( M \) is allowed to slide without friction along a horizontal tabletop. Connected to the left of this cart is an ideal spring of stiffness constant \( k \), which is also connected to a fixed support to the left of the spring. A rigid rod of mass \( m \) is allowed to pivot without friction about the center of the cart as shown in the figure below.

Using the generalized coordinates \( q_1 = x \) (as measured along the tabletop from where the spring is at its natural length) and \( q_2 = \theta \) is the angle shown in the figure above, we see that the kinetic energy of the system is given by

\[
T = T_{\text{Cart}} + T_{\text{Rod}}
\]

where

\[
T_{\text{Cart}} = \frac{1}{2} M \dot{x}^2
\]

\((M \text{ being the mass of the cart})\) and

\[
T_{\text{Rod}} = \frac{1}{2} m (\dot{x}_{\text{cm}}^2 + \dot{y}_{\text{cm}}^2) + \frac{1}{2} I_{\text{cm}} \dot{\theta}^2.
\]

But

\[
x_{\text{cm}} = x + \frac{L}{2} \sin(\theta) \quad \text{and} \quad y_{\text{cm}} = -\frac{L}{2} \cos(\theta)
\]
\[ \dot{x}_{cm} = \dot{x} + \frac{L}{2} \dot{\theta} \cos(\theta) \quad \text{and} \quad \dot{y}_{cm} = \frac{L}{2} \dot{\theta} \sin(\theta) \]

and
\[ I_{cm} = \frac{1}{12} m L^2 \]

where \( L \) is the length of the rod and \( m \) is the mass of the rod. Putting these into the expression for \( T_{\text{Rod}} \), we get
\[ T_{\text{Rod}} = \frac{1}{2} m \left\{ \left( \dot{x} + \frac{L}{2} \dot{\theta} \cos(\theta) \right)^2 + \left( \frac{L}{2} \dot{\theta} \sin(\theta) \right)^2 \right\} + \frac{1}{2} \left( \frac{1}{12} m L^2 \right) \dot{\theta}^2 \]

which reduces to
\[ T_{\text{Rod}} = \frac{1}{2} m \left\{ \dot{x}^2 + L \dot{x} \dot{\theta} \cos(\theta) + \frac{L^2}{4} \dot{\theta}^2 \right\} + \frac{1}{24} m L^2 \dot{\theta}^2 \]

or simply
\[ T_{\text{Rod}} = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m L \dot{x} \dot{\theta} \cos(\theta) + \frac{1}{6} m L^2 \dot{\theta}^2. \]

Thus we see that the total kinetic energy of the cart-rod system is given by
\[ T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m L \dot{x} \dot{\theta} \cos(\theta) + \frac{1}{6} m L^2 \dot{\theta}^2 \]

or simply
\[ T = \frac{1}{2} (m + M) \dot{x}^2 + \frac{1}{2} m L \dot{x} \dot{\theta} \cos(\theta) + \frac{1}{6} m L^2 \dot{\theta}^2. \]

In addition, we the total potential energy of the cart-rod system is given by
\[ V = \frac{1}{2} k x^2 + m g y_{cm} = \frac{1}{2} k x^2 - \frac{1}{2} m g L \cos(\theta), \]

using the pivot point \( O \) as the zero point for weight potential energy. From these, we can now construct the Lagrangian for the system as \( \mathcal{L} = T - V \) or
\[ \mathcal{L} = \frac{1}{2} (m + M) \dot{x}^2 + \frac{1}{2} m L \dot{x} \dot{\theta} \cos(\theta) + \frac{1}{6} m L^2 \dot{\theta}^2 - \frac{1}{2} k x^2 + \frac{1}{2} m g L \cos(\theta). \]

Then
\[ \frac{\partial \mathcal{L}}{\partial \dot{x}} = (m + M) \dot{x} + \frac{1}{2} m L \dot{\theta} \cos(\theta) \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial x} = -k x \]
and hence
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0 \]
leads to
\[ \frac{d}{dt} \left( (m + M)\dot{x} + \frac{1}{2} mL\dot{\theta} \cos(\theta) \right) + kx = 0 \]
or
\[ (m + M)\ddot{x} + \frac{1}{2} mL\dot{\theta} \cos(\theta) - \frac{1}{2} mL\dot{\theta}^2 \sin(\theta) + kx = 0 \]
as one of the equations of motion for the cart-rod system. We also have
\[ \frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2} mL\dot{x} \cos(\theta) + \frac{1}{3} mL^2\dot{\theta}, \quad \frac{\partial L}{\partial \theta} = -\frac{1}{2} mL\dot{x}\dot{\theta} \sin(\theta) - \frac{1}{2} mgL \sin(\theta) \]
and hence
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0 \]
leads to
\[ \frac{d}{dt} \left( \frac{1}{2} mL\dot{x} \cos(\theta) + \frac{1}{3} mL^2\dot{\theta} \right) + \frac{1}{2} mL\dot{x}\dot{\theta} \cos(\theta) + \frac{1}{2} mgL \sin(\theta) = 0 \]
or
\[ \frac{1}{2} mL\ddot{x} \cos(\theta) - \frac{1}{2} mL\dot{x}\dot{\theta} \sin(\theta) + \frac{1}{3} mL^2\ddot{\theta} + \frac{1}{2} mL\dot{x}\dot{\theta} \sin(\theta) + \frac{1}{2} mgL \sin(\theta) = 0 \]
which reduces to
\[ \frac{1}{2} mL\ddot{x} \cos(\theta) + \frac{1}{3} mL^2\ddot{\theta} + \frac{1}{2} mgL \sin(\theta) = 0 \]
as the second equation of motion for the cart-rod system.

**Stability Analysis About The Equilibrium Points for This Example**

From the expression for \( V \) we see that
\[ \frac{\partial V}{\partial x} = kx = 0 \quad \text{and} \quad \frac{\partial V}{\partial \theta} = \frac{1}{2} mgL \sin(\theta) = 0 \]
give the equilibrium points at \((x_e, \theta_e) = (0, 0)\) and at \((x_e, \theta_e) = (0, \pi)\). It is interesting to consider small oscillations about the equilibrium at \(x_e = 0, \theta_e = 0\), for this problem by replacing
\[
\cos(\theta) = \cos(\varepsilon_\theta + \theta_e) = \cos(\varepsilon_\theta) \simeq 1
\]
and
\[
\sin(\theta) = \sin(\varepsilon_\theta + \theta_e) = \sin(\varepsilon_\theta) \simeq \varepsilon_\theta
\]
thereby removing all orders higher than one. Under these assumptions, the equations of motion
\[
(m + M)\ddot{x} + \frac{1}{2} mL\ddot{\theta}\cos(\theta) - \frac{1}{2} mL\dot{\theta}^2 \sin(\theta) + kx = 0
\]
and
\[
\frac{1}{2} mL\ddot{x}\cos(\theta) + \frac{1}{3} mL^2\dot{\theta} + \frac{1}{2} mgL\sin(\theta) = 0
\]
(to first order) become
\[
(m + M)\dddot{x} + \frac{1}{2} mL\dddot{\theta} + k\dddot{x} = 0
\]
and
\[
\frac{1}{2} mL\dddot{x} + \frac{1}{3} mL^2\dddot{\theta} + \frac{1}{2} mgL\dddot{\theta} = 0
\]
where \(\varepsilon_x = x - x_e\) and \(\varepsilon_\theta = \theta - \theta_e\). These equation reduce to
\[
2(m + M)\dddot{x} + mL\dddot{\theta} + 2k\varepsilon_x = 0
\]
and
\[
3\dddot{x} + 2L\dddot{\theta} + 3g\varepsilon_\theta = 0
\]
which in matrix form can be expressed as
\[
\begin{bmatrix}
2(m + M) & mL \\
3 & 2L
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_\theta
\end{bmatrix}
+ \begin{bmatrix}
2k & 0 \\
0 & 3g
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_\theta
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}.
\]
By setting
\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_\theta
\end{bmatrix}
= \begin{bmatrix}
A_1 \\
A_2
\end{bmatrix} e^{i\omega t} \neq \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]
in this equation we get

\[-\omega^2\begin{bmatrix} \frac{2(m + M)}{3} & mL \\ \frac{mL}{2L} & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} e^{i\omega t} + \begin{bmatrix} 2k \\ 0 \\ 0 \\ 3g \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} e^{i\omega t} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\]

or

\[-\omega^2\begin{bmatrix} \frac{2(m + M)}{3} & mL \\ \frac{mL}{2L} & 0 \end{bmatrix} + \begin{bmatrix} 2k \\ 0 \\ 0 \\ 3g \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\]

or

\[\begin{bmatrix} 2k - 2\omega^2(m + M) & -mL\omega^2 \\ -3\omega^2 & 3g - 2L\omega^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\]

in order for

\[\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}\]

we must require that

\[\det\begin{bmatrix} 2k - 2\omega^2(m + M) & -mL\omega^2 \\ -3\omega^2 & 3g - 2L\omega^2 \end{bmatrix} = 0\]

and this leads to

\[(m + 4M)L\omega^4 - 2(2kL + 3(m + M)g)\omega^2 + 6kg = 0\]

so that the natural angular frequencies of the cart-rod system are given by

\[\omega^2 = \frac{2(2kL + 3(m + M)g) \pm \sqrt{4(2kL + 3(m + M)g)^2 - 24kgL(m + 4M)}}{2(m + 4M)L}\]

or simply

\[\omega^2 = \frac{2kL + 3(m + M)g}{(m + 4M)L} \pm \sqrt{\left(\frac{2kL + 3(m + M)g}{(m + 4M)L}\right)^2 - \frac{6kg}{(m + 4M)L}}\]

To check this result, let us consider some limiting cases. The first is when \(L = 0\). This leads to

\[(m + 4M)L\omega^4 - 2(2kL + 3(m + M)g)\omega^2 + 6kg = 0\]

which for \(L = 0\) reduces to simply

\[-6(m + M)g\omega^2 + 6kg = 0\]
which leads to
\[ \omega = \sqrt{\frac{k}{m + M}}, \]
which is expected, since we may treat this cart-rod system as a simple block of mass \( m + M \) connected to a spring of constant \( k \). The next limiting case is when \( M \to \infty \). This leads to
\[ \omega^2 = \frac{3g}{4L} \pm \sqrt{\left(\frac{3g}{4L}\right)^2}, \]
or\[ \omega^* = \sqrt{\frac{3g}{2L}}, \]
and \( \omega_- = 0 \), which corresponds to no motion, and hence we ignore this.

The result for \( \omega^* \) is expected, since we may treat the system as a compound pendulum with moment of inertia
\[ I_O = \frac{1}{3} m L^2, \]
about point O on the cart (which is not moving since \( M \to \infty \)). The center-of-mass of the rod is a distance \( D = L/2 \) from point O. Using the result
\[ \omega = \sqrt{\frac{mgD}{I_O}} \]
from basic physics and putting in our expressions for \( D \) and \( I_O \), we then get
\[ \omega = \sqrt{\frac{mg(L/2)}{mL^2/3}} = \sqrt{\frac{3g}{2L}} \]
which agrees with \( \omega^* \) above.

Let us also consider small oscillations about the equilibrium point at \( x_e = 0 \), \( \theta_e = \pi \), for this problem by replacing
\[ \cos(\theta) = \cos(\varepsilon_\theta + \pi) = -\cos(\varepsilon_\theta) \approx -1 \]
and
\[ \sin(\theta) = \sin(\varepsilon_\theta + \pi) = -\sin(\varepsilon_\theta) \approx -\varepsilon_\theta \]
thereby removing all orders higher than one. Under these assumptions, the equations of motion,

$$(m + M)\ddot{x} + \frac{1}{2}mL\ddot{\theta}\cos(\theta) - \frac{1}{2}mL\dot{\theta}^2\sin(\theta) + kx = 0$$

and

$$\frac{1}{2}mL\ddot{x}\cos(\theta) + \frac{1}{3}mL^2\ddot{\theta} + \frac{1}{2}mgL\sin(\theta) = 0$$

to first order become

$$(m + M)\ddot{\epsilon}_x - \frac{1}{2}mL\ddot{\epsilon}_\theta + k\ddot{\epsilon}_x = 0$$

and

$$-\frac{1}{2}mL\dddot{\epsilon}_x + \frac{1}{3}mL^2\dddot{\epsilon}_\theta - \frac{1}{2}mgL\dddot{\epsilon}_\theta = 0$$

where $\epsilon_x = x - x_e$ and $\epsilon_\theta = \theta - \theta_e$. These equation reduce to

$$2(m + M)\ddot{\epsilon}_x - mL\dddot{\epsilon}_\theta + 2k\ddot{\epsilon}_x = 0$$

and

$$3\dddot{\epsilon}_x - 2L\dddot{\epsilon}_\theta + 3g\dddot{\epsilon}_\theta = 0$$

which in matrix form can be expressed as

$$\begin{bmatrix} 2(m + M) & -mL \\ 3 & -2L \end{bmatrix} \begin{bmatrix} \ddot{\epsilon}_x \\ \ddot{\epsilon}_\theta \end{bmatrix} + \begin{bmatrix} 2k & 0 \\ 0 & 3g \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_\theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$  

By setting

$$\begin{bmatrix} \epsilon_x \\ \epsilon_\theta \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} e^{i\omega t} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

in this equation we get

$$-\omega^2 \begin{bmatrix} 2(m + M) & -mL \\ 3 & -2L \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} e^{i\omega t} + \begin{bmatrix} 2k & 0 \\ 0 & 3g \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} e^{i\omega t} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$  

or

$$\begin{bmatrix} -\omega^2 & 2(m + M) & -mL \\ 3 & -2L \\ 2k & 0 & 3g \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$  

or

$$\begin{bmatrix} 2k - 2\omega^2(m + M) & mL\omega^2 \\ -3\omega^2 & 3g + 2L\omega^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$  

8
in order for
\[
\begin{pmatrix}
A_1 \\
A_2
\end{pmatrix} \neq \begin{pmatrix} 0 \\
0
\end{pmatrix}
\]
we must require that
\[
\det \begin{bmatrix}
2k - 2\omega^2(m + M) & mL\omega^2 \\
-3\omega^2 & 3g + 2L\omega^2
\end{bmatrix} = 0
\]
and this leads to
\[
(m + 4M)L\omega^4 - 2(2kL - 3(m + M)g)\omega^2 - 6kg = 0
\]
so that the natural angular frequencies of the cart-rod system are given by
\[
\omega^2_\pm = \frac{2(2kL + 3(m + M)g) \pm \sqrt{4(2kL + 3(m + M)g)^2 + 24kgL(m + 4M)}}{2(m + 4M)L}.
\]
or simply
\[
\omega^2_\pm = \frac{2kL + 3(m + M)g \pm \sqrt{\left(\frac{2kL + 3(m + M)g}{m + 4M}\right)^2 + \frac{6kg}{(m + 4M)L}}}{(m + 4M)L}.
\]
Note then that \(\omega^2_- < 0\) showing that \(\omega_-\) is not real and so
\[
\begin{pmatrix} \varepsilon_x \\
\varepsilon_\theta
\end{pmatrix} = \begin{pmatrix} A_1 \\
A_2
\end{pmatrix} e^{i\omega t}
\]
will not be bounded, and hence this equilibrium point is unstable.

3. Example: The Spherical Pendulum

The figure below shows a spherical pendulum which consists of a mass \(m\) connected to a pivot point \(O\) located at the point \((0,0,L)\) by a cord of fixed
Let $\theta$ be the polar angle that the cords makes with the negative vertical $z$ axis and let $\varphi$ be the azimuthal angle that the line connecting the projection of the mass on the $xy$ plane and the origin (the dotted line in the figure above) makes with the positive $x$ axis. Using the generalized coordinates: $q_1 = \theta$ and $q_2 = \varphi$, we see that the $xyz$ position of the mass is given by

$$x = L \sin(\pi - \theta) \cos(\varphi) = L \sin(\theta) \cos(\varphi)$$

and

$$y = L \sin(\pi - \theta) \sin(\varphi) = L \sin(\theta) \sin(\varphi)$$

and

$$z = L - L \cos(\pi - \theta) = L + L \cos(\theta).$$

Thus we have

$$\dot{x} = L \dot{\theta} \cos(\theta) \cos(\varphi) - L \dot{\varphi} \sin(\theta) \sin(\varphi)$$

and

$$\dot{y} = L \dot{\theta} \cos(\theta) \sin(\varphi) + L \dot{\varphi} \sin(\theta) \cos(\varphi)$$

and

$$\dot{z} = -L \dot{\theta} \sin(\theta).$$

The kinetic energy of the pendulum is then given by

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$
which reduces to
\[ T = \frac{1}{2} m((L\dot{\theta} \cos(\theta) \cos(\varphi) - L\dot{\varphi} \sin(\theta) \sin(\varphi))^2 + \frac{1}{2} m(L\dot{\theta} \cos(\theta) \sin(\varphi) + L\dot{\varphi} \sin(\theta) \cos(\varphi))^2 + \frac{1}{2} m(-L\dot{\theta} \sin(\theta))^2 \]
or
\[ T = \frac{1}{2} m((L\dot{\theta})^2 \cos^2(\theta) \cos^2(\varphi) + (L\dot{\varphi})^2 \sin^2(\theta) \sin^2(\varphi)) + \frac{1}{2} m((L\dot{\theta})^2 \cos^2(\theta) \sin^2(\varphi) + (L\dot{\varphi})^2 \sin^2(\theta) \cos^2(\varphi)) + \frac{1}{2} m(L\dot{\theta})^2 \sin^2(\theta) \]
which further reduces to
\[ T = \frac{1}{2} mL^2(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2(\theta)). \]

The potential energy is given by
\[ V = mgz = mgL(1 + \cos(\theta)) \]
so that the Lagrangian of the system is \( \mathcal{L} = T - V \), or
\[ \mathcal{L} = \frac{1}{2} mL^2(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2(\theta)) - mgL(1 + \cos(\theta)). \]

Then we see that
\[ \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mL^2 \dot{\theta} \]
and
\[ \frac{\partial \mathcal{L}}{\partial \theta} = mL^2 \dot{\varphi}^2 \sin(\theta) \cos(\theta) + mgL \sin(\theta) \]
resulting in the one equation of motion
\[ \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \]
or
\[ mL^2 \ddot{\theta} - mL^2 \dot{\varphi}^2 \sin(\theta) \cos(\theta) - mgL \sin(\theta) = 0, \]
which reduces to
\[ \ddot{\theta} - \dot{\varphi}^2 \sin(\theta) \cos(\theta) - \frac{g}{L} \sin(\theta) = 0. \]
Next we see that
\[
\frac{\partial L}{\partial \dot{\phi}} = mL^2 \dot{\phi} \sin^2(\theta) \quad , \quad \frac{\partial L}{\partial \phi} = 0
\]
resulting in the second equation of motion
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) = \frac{d}{dt} \left( mL^2 \dot{\phi} \sin^2(\theta) \right) = 0
\]
which says that \( mL^2 \dot{\phi} \sin^2(\theta) \) equals a constant in time. Setting this equal to \( C \), we get
\[
\dot{\phi} = \frac{C}{mL^2 \sin^2(\theta)}
\]
and putting this into the first equation of motion leads to
\[
\ddot{\theta} - \left( \frac{C}{mL^2 \sin^2(\theta)} \right)^2 \sin(\theta) \cos(\theta) - \frac{g}{L} \sin(\theta) = 0
\]
or simply
\[
\ddot{\theta} - \frac{C^2 \cos(\theta)}{m^2 L^4 \sin^3(\theta)} - \frac{g}{L} \sin(\theta) = 0.
\]
As a check, we note that if there is no motion in \( \phi \) direction so that \( \dot{\phi} = 0 \), then \( C = 0 \) and the \( \theta \) equation becomes
\[
\ddot{\theta} - \frac{g}{L} \sin(\theta) = 0
\]
which leads to a stable equilibrium at \( \theta = \pi \), and then
\[
\sin(\theta) \simeq \sin(\pi) + (\theta - \pi) \cos(\pi) = -(\theta - \pi)
\]
and the equation of motion becomes
\[
\ddot{\epsilon} + \frac{g}{L} (\theta - \pi) = 0.
\]
Setting \( \epsilon = \theta - \pi \), we then get
\[
\ddot{\epsilon} + \frac{g}{L} \epsilon = 0
\]
which is the equation of a simple pendulum with angular frequency

$$\omega = \sqrt{\frac{g}{L}}.$$ 

We also note from the equation of motion

$$\ddot{\theta} - \frac{C^2 \cos(\theta)}{m^2 L^4 \sin^3(\theta)} - \frac{g}{L} \sin(\theta) = 0.$$ 

that

$$\ddot{\theta} = \frac{C^2 \cos(\theta)}{m^2 L^4 \sin^3(\theta)} + \frac{g}{L} \sin(\theta) \equiv G(\theta)$$

so that

$$\dot{\theta} \ddot{\theta} = \dot{\theta} G(\theta) \quad \text{or} \quad \dot{\theta} d\dot{\theta} = G(\theta) d\theta$$

or

$$\frac{1}{2} \dot{\theta}^2 - \int G(\theta) d\theta = \text{constant}$$

in time. This leads to

$$\frac{1}{2} \dot{\theta}^2 - \int \left( \frac{C^2 \cos(\theta)}{m^2 L^4 \sin^3(\theta)} + \frac{g}{L} \sin(\theta) \right) d\theta = \text{constant}$$

or

$$\frac{1}{2} \dot{\theta}^2 + \frac{C^2}{2m^2 L^4 \sin^2(\theta)} + \frac{g}{L} \cos(\theta) = \text{constant}$$

or

$$\frac{1}{2} \dot{\theta}^2 + \frac{1}{2} \left( \frac{C}{m L^2 \sin^2(\theta)} \right)^2 \sin^2(\theta) + \frac{g}{L} \cos(\theta) = \text{constant}$$

Of course this is simply

$$\frac{1}{2} m L^2 (\dot{\theta}^2 + \dot{\varphi}^2 \sin^2(\theta)) + mg \cos(\theta) = \text{constant}.$$ 

or

$$T + V = \text{constant}$$

and thereby showing that energy is conserved.
4. The Equations of Motion - General Form

Consider a system of \( N \) particles with positions given by the rectangular co-
ordinates \((x_i, y_i, z_i)\), for \( i = 1, 2, 3, \ldots, N \), and having \( n \) degrees of freedom with
generalized coordinates \( q_1, q_2, q_3, \ldots, q_n \) and generalized speeds \( \dot{q}_1, \dot{q}_2, \dot{q}_3, \ldots, \dot{q}_n \),
so that

\[
\begin{align*}
x_i &= x_i(q_1, q_2, q_3, \ldots, q_n) \\
y_i &= y_i(q_1, q_2, q_3, \ldots, q_n) \\
z_i &= z_i(q_1, q_2, q_3, \ldots, q_n)
\end{align*}
\]

for \( i = 1, 2, 3, \ldots, N \). We know that the Lagrangian of the system can be written as

\[
\mathcal{L} = T - V
\]

and in general, we shall always find that the kinetic energy of the system will have the form

\[
T = \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} T_{jk} \dot{q}_j \dot{q}_k
\]

(5a)

with

\[
T_{jk} \equiv \sum_{i=1}^{N} m_i \left\{ \frac{\partial x_i}{\partial q_j} \frac{\partial x_i}{\partial q_k} + \frac{\partial y_i}{\partial q_j} \frac{\partial y_i}{\partial q_k} + \frac{\partial z_i}{\partial q_j} \frac{\partial z_i}{\partial q_k} \right\}
\]

(5b)

being only a function of the \( q_k \)'s (not the \( \dot{q}_k \)'s). Thus we have

\[
\mathcal{L} = \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} T_{jk} \dot{q}_j \dot{q}_k - V,
\]

(6)

and it should be clear from Equation (5a) that

\[
T_{jk} \equiv \frac{\partial^2 T}{\partial \dot{q}_j \partial \dot{q}_k}
\]

(7)

and so \( T_{jk} = T_{kj} \). Now all of \( T_{jk} \) and \( V \) are functions of only the generalized
coordinates \( q_j \) and they are not functions of the generalized speeds \( \dot{q}_j \). From
Lagrange’s equations

\[
\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0
\]

"
we have
\[ \frac{d}{dt} \left( \frac{\partial (T - V)}{\partial \dot{q}_i} \right) - \frac{\partial (T - V)}{\partial q_i} = 0 \]
or
\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial \dot{q}_i} = 0. \] (8)

Since
\[ T = \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} T_{jk} \dot{q}_j \dot{q}_k \]
we see that
\[ \frac{\partial T}{\partial q_i} = \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial T_{jk}}{\partial q_i} \dot{q}_j \dot{q}_k. \] (9a)

In addition we see that
\[ \frac{\partial T}{\partial \dot{q}_i} = \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial T_{jk}}{\partial \dot{q}_i} \dot{q}_j \dot{q}_k. \]

But
\[ \frac{\partial \dot{q}_j}{\partial q_i} = \begin{cases} 0, & \text{for } j \neq i \\ 1, & \text{for } j = i \end{cases} = \delta_{ji} \]
and
\[ \frac{\partial \dot{q}_k}{\partial q_i} = \begin{cases} 0, & \text{for } k \neq i \\ 1, & \text{for } k = i \end{cases} = \delta_{ki} \]
and so we find that
\[ \frac{\partial T}{\partial \dot{q}_i} = \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} T_{jk} (\delta_{ji} \dot{q}_k + \delta_{ki} \dot{q}_j) = \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} T_{jk} \delta_{ji} \dot{q}_k + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} T_{jk} \delta_{ki} \dot{q}_j \]
\[ = \frac{1}{2} \sum_{k=1}^{n} T_{ik} \dot{q}_k + \frac{1}{2} \sum_{j=1}^{n} T_{ji} \dot{q}_j = \frac{1}{2} \sum_{j=1}^{n} T_{ij} \dot{q}_j + \frac{1}{2} \sum_{j=1}^{n} T_{ji} \dot{q}_j \]
\[ = \frac{1}{2} \sum_{j=1}^{n} (T_{ij} + T_{ji}) \dot{q}_j = \frac{1}{2} \sum_{j=1}^{n} (T_{ij} + T_{ij}) \dot{q}_j \]

since \( T_{ij} = T_{ji} \), and so we find that
\[ \frac{\partial T}{\partial \dot{q}_i} = \sum_{j=1}^{n} T_{ij} \dot{q}_j. \] (9b)
Putting Equations (9a) and (9b) into Equation (8), we get

\[
\frac{d}{dt}\left(\sum_{j=1}^{n} T_{ij} \dot{q}_j\right) - \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial T_{jk}}{\partial q_i} \dot{q}_j \dot{q}_k + \frac{\partial V}{\partial q_i} = 0
\]

or

\[
\left(\sum_{j=1}^{n} \frac{d}{dt}(T_{ij} \dot{q}_j)\right) - \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial T_{jk}}{\partial q_i} \dot{q}_j \dot{q}_k + \frac{\partial V}{\partial q_i} = 0.
\]  

(10)

But

\[
\sum_{j=1}^{n} \frac{d}{dt}(T_{ij} \dot{q}_j) = \sum_{j=1}^{n} \left(\frac{dT_{ij}}{dt} \dot{q}_j + T_{ij} \frac{d\dot{q}_j}{dt}\right) = \sum_{j=1}^{n} \left(\frac{dT_{ij}}{dt} \dot{q}_j + T_{ij} \ddot{q}_j\right).
\]

However, using the chain rule from Calculus, we have

\[
\frac{dT_{ij}}{dt} = \sum_{k=1}^{n} \frac{\partial T_{ij}}{\partial q_k} \frac{dq_k}{dt} = \sum_{k=1}^{n} \frac{\partial T_{ij}}{\partial q_k} \dot{q}_k
\]

and so

\[
\sum_{j=1}^{n} \frac{d}{dt}(T_{ij} \dot{q}_j) = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial T_{ij}}{\partial q_k} \dot{q}_k \dot{q}_j + \sum_{j=1}^{n} T_{ij} \ddot{q}_j
\]

and putting this into Equation (10), we have

\[
\sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial T_{ij}}{\partial q_k} \dot{q}_k \dot{q}_j + \sum_{j=1}^{n} T_{ij} \ddot{q}_j - \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial T_{jk}}{\partial q_i} \dot{q}_j \dot{q}_k + \frac{\partial V}{\partial q_i} = 0
\]

or just

\[
\sum_{j=1}^{n} T_{ij} \ddot{q}_j + \sum_{j=1}^{n} \sum_{k=1}^{n} \left(\frac{\partial T_{ij}}{\partial q_k} - \frac{1}{2} \frac{\partial T_{jk}}{\partial q_i}\right) \dot{q}_j \dot{q}_k + \frac{\partial V}{\partial q_i} = 0
\]

Since \( T_{ij} = T_{ji} \), we may write

\[
\frac{\partial T_{ij}}{\partial q_k} = \frac{1}{2} \frac{\partial T_{ij}}{\partial q_k} + \frac{1}{2} \frac{\partial T_{ji}}{\partial q_k}
\]

and then we have

\[
\sum_{j=1}^{n} T_{ij} \ddot{q}_j + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \left(\frac{\partial T_{ij}}{\partial q_k} + \frac{\partial T_{ji}}{\partial q_k} - \frac{\partial T_{jk}}{\partial q_i}\right) \dot{q}_j \dot{q}_k + \frac{\partial V}{\partial q_i} = 0.
\]  

(11)
or
\[ \sum_{j=1}^{n} T_{ij} \ddot{q}_j + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} T_{ijk} \dot{q}_j \dot{q}_k + \frac{\partial V}{\partial q_i} = 0. \] (12a)
with
\[ T_{ij} = \left. \frac{\partial^2 T}{\partial \dot{q}_j \partial \dot{q}_k} \right|_{(q_1, q_2, q_3, \ldots, q_n) = (q_{ie}, q_{ie}, q_{ie}, \ldots, q_{ie})} \] (12b)
and
\[ T_{ijk} = \left. \frac{\partial T_{ij}}{\partial q_k} + \frac{\partial T_{ji}}{\partial q_k} - \frac{\partial T_{jk}}{\partial q_i} \right|_{(q_1, q_2, q_3, \ldots, q_n) = (q_{ie}, q_{ie}, q_{ie}, \ldots, q_{ie})}. \] (12c)
Equation (12) give the equations of motion for the n-degree of freedom system and it clearly shows the explicit occurrences of the \( \ddot{q}_j \)'s and the \( \dot{q}_j \)'s.

5. Linear Stability Analysis About Static Equilibrium Points

The static equilibrium points for the system are where \( q_i = q_{ie} = \text{constant} \), resulting in \( \ddot{q}_i = \dot{q}_i = 0 \) and putting this into Equation (12a), we find that
\[ \frac{\partial V}{\partial q_j} \bigg|_{(q_1, q_2, q_3, \ldots, q_n) = (q_{ie}, q_{ie}, q_{ie}, \ldots, q_{ie})} = 0 \] (13)
for \( j = 1, 2, 3, \ldots, n \) gives the set of equations that can be solved for the \( q_{ie} \)'s.

Now suppose that
\[ (q_1, q_2, q_3, \ldots, q_n) = (q_{1e}, q_{2e}, q_{3e}, \ldots, q_{ne}). \]
is one such static equilibrium point and suppose that
\[ (\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n) = (q_1 - q_{1e}, q_2 - q_{2e}, q_3 - q_{3e}, \ldots, q_n - q_{ne}) \]
is a small perturbation about this static equilibrium point. Using the notation
\[ V = V(q_1, q_2, q_3, \ldots, q_n) \quad \text{and} \quad V_e = V(q_{1e}, q_{2e}, q_{3e}, \ldots, q_{ne}) \]
and expanding \( \partial V/\partial q_i \) to first order, we have
\[ \frac{\partial V}{\partial q_i} = \left. \frac{\partial V}{\partial q_i} \right|_{(q_1, q_2, q_3, \ldots, q_n) = (q_{1e}, q_{2e}, q_{3e}, \ldots, q_{ne})} + \sum_{j=1}^{n} \frac{\partial^2 V}{\partial q_i \partial \dot{q}_j} \bigg|_{(q_1, q_2, q_3, \ldots, q_n) = (q_{1e}, q_{2e}, q_{3e}, \ldots, q_{ne})} (q_j - q_{je}). \]
or
\[ \frac{\partial V}{\partial q_i} = \sum_{j=1}^{n} \frac{\partial^2 V}{\partial q_i \partial q_j} (q_j - q_{je}) \]
since
\[ \frac{\partial V}{\partial q_i} |_{(q_1, q_2, \ldots, q_n) = (q_{1e}, q_{2e}, \ldots, q_{ne})} = 0. \]

If we define
\[ V_{ije} = \frac{\partial^2 V}{\partial q_i \partial q_j} \bigg|_{(q_1, q_2, \ldots, q_n) = (q_{1e}, q_{2e}, \ldots, q_{ne})} \]
then we get
\[ \frac{\partial V}{\partial q_i} = \sum_{j=1}^{n} V_{ije} (q_j - q_{je}) = \sum_{j=1}^{n} V_{ije} \varepsilon_j \]
to first order in the \( \varepsilon_j \)'s. Setting \( q_j = q_{je} \) in the \( T_{ij} \)'s and the \( T_{ijk} \)'s, we have
\[ T_{ije} \equiv T_{ij}(q_{1e}, q_{2e}, q_{3e}, \ldots, q_{ne}) = \frac{\partial^2 T}{\partial q_i \partial q_j} \bigg|_{(q_1, q_2, q_3, \ldots, q_n) = (q_{1e}, q_{2e}, q_{3e}, \ldots, q_{ne})} \]
and
\[ T_{ijke} \equiv \frac{\partial T_{ij}}{\partial q_k} + \frac{\partial T_{jk}}{\partial q_i} - \frac{\partial T_{ik}}{\partial q_j} \bigg|_{(q_1, q_2, q_3, \ldots, q_n) = (q_{1e}, q_{2e}, q_{3e}, \ldots, q_{ne})} \]
Putting these into Equation (12), we get
\[ \sum_{j=1}^{n} T_{ije} \ddot{q}_j + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} T_{ijke} \dot{q}_j \dot{q}_k + \sum_{j=1}^{n} V_{ije} \varepsilon_j = 0 \]
or
\[ \sum_{j=1}^{n} T_{ije} \ddot{\varepsilon}_j + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} T_{ijke} \dot{\varepsilon}_j \dot{\varepsilon}_k + \sum_{j=1}^{n} V_{ije} \varepsilon_j = 0 \]
and since the middle term in this equation is second order in the \( \varepsilon_j \)'s, we may drop it and simply write
\[ \sum_{j=1}^{n} T_{ije} \ddot{\varepsilon}_j + \sum_{j=1}^{n} V_{ije} \varepsilon_j = 0. \]

Writing this in matrix form, we have
\[ [T_{e}]_{n \times n} \{\ddot{\varepsilon}\}_{n \times 1} + [V_{e}]_{n \times n} \{\varepsilon\}_{n \times 1} = \{0\}_{n \times 1} \] (18a)
where
\[ [T_e]_{n \times n} = \begin{bmatrix} \frac{\partial^2 T}{\partial q_i \partial q_j} \bigg|_{(q_1, q_2, q_3, \ldots, q_n) = (q_{1e}, q_{2e}, q_{3e}, \ldots, q_{ne})} \end{bmatrix} \]  
(18b)

and
\[ [V_e]_{n \times n} = \begin{bmatrix} \frac{\partial^2 V}{\partial q_i \partial q_j} \bigg|_{(q_1, q_2, q_3, \ldots, q_n) = (q_{1e}, q_{2e}, q_{3e}, \ldots, q_{ne})} \end{bmatrix}. \]  
(18c)

To examine the stability of the equilibrium point at
\[ (q_1, q_2, q_3, \ldots, q_n) = (q_{1e}, q_{2e}, q_{3e}, \ldots, q_{ne}) \]
we set
\[ \{ \varepsilon \}_{n \times 1} = \{ \mu \}_{n \times 1} e^{j\omega t} \neq \{ 0 \}_{n \times 1} \]
into Equation (18a) resulting in
\[ -\omega^2 [T_e]_{n \times n} \{ \mu \}_{n \times 1} e^{j\omega t} + [V_e]_{n \times n} \{ \mu \}_{n \times 1} e^{j\omega t} = \{ 0 \}_{n \times 1} \]
or
\[ (-\omega^2 [T_e]_{n \times n} + [V_e]_{n \times n}) \{ \mu \}_{n \times 1} = \{ 0 \}_{n \times 1}. \]

In order for \( \{ \mu \}_{n \times 1} \neq \{ 0 \}_{n \times 1} \), we must have
\[ \det(-\omega^2 [T_e]_{n \times n} + [V_e]_{n \times n}) = 0 \]  
(19)
which shows that if all solutions to the equation are real, the equilibrium point at
\[ (q_1, q_2, q_3, \ldots, q_n) = (q_{1e}, q_{2e}, q_{3e}, \ldots, q_{ne}) \]
is stable and if at least one solution to Equation (20) is not real, then the equilibrium point at
\[ (q_1, q_2, q_3, \ldots, q_n) = (q_{1e}, q_{2e}, q_{3e}, \ldots, q_{ne}) \]
is unstable. If both \([T_e]_{n \times n}\) and \([V_e]_{n \times n}\) are zero, then higher-order analysis is necessary.
6. The Cart-Rod Example: Revisited

Consider the cart-rod system discussed earlier. We saw that the total kinetic energy of the cart-rod system is given by

\[ T = \frac{1}{2}(m + M)\dot{x}^2 + \frac{1}{2}mL\dot{x}\dot{\theta}\cos(\theta) + \frac{1}{6}mL^2\dot{\theta}^2. \]

In addition, we saw that the total potential energy of the cart-rod system was given by

\[ V = \frac{1}{2}kx^2 - \frac{1}{2}mgL\cos(\theta), \]

From these, we can now construct

\[ \left\{ \begin{array}{c} \frac{\partial V}{\partial x} \\ \frac{\partial V}{\partial \theta} \end{array} \right\} = \left\{ \begin{array}{c} kx \\ \frac{1}{2}mgL\sin(\theta) \end{array} \right\} \]

and setting this equal to zero gives the equilibrium points at \((x_e, \theta_e) = (0, 0)\) and at \((x_e, \theta_e) = (0, \pi)\). Next we have

\[ [T] = \begin{bmatrix} \frac{\partial^2 T}{\partial \dot{x}\partial \dot{x}} & \frac{\partial^2 T}{\partial \dot{x}\partial \dot{\theta}} \\ \frac{\partial^2 T}{\partial \dot{x}\partial \dot{\theta}} & \frac{\partial^2 T}{\partial \dot{\theta}\partial \dot{\theta}} \end{bmatrix} = \begin{bmatrix} m + M & \frac{1}{2}mL\cos(\theta) \\ \frac{1}{2}mL\cos(\theta) & \frac{1}{3}mL^2 \end{bmatrix} \]

and

\[ [V] = \begin{bmatrix} \frac{\partial^2 V}{\partial x\partial x} & \frac{\partial^2 V}{\partial x\partial \theta} \\ \frac{\partial^2 V}{\partial x\partial \theta} & \frac{\partial^2 V}{\partial \theta\partial \theta} \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & \frac{1}{2}mgL\cos(\theta) \end{bmatrix} \]

and so for \((x_e, \theta_e) = (0, 0)\), we have

\[ [T_e] = \begin{bmatrix} m + M & \frac{1}{2}mL \\ \frac{1}{2}mL & \frac{1}{3}mL^2 \end{bmatrix} \quad \text{and} \quad [V_e] = \begin{bmatrix} k & 0 \\ 0 & \frac{1}{2}mgL \end{bmatrix} \]

and for \((x_e, \theta_e) = (0, \pi)\), we have

\[ [T_e] = \begin{bmatrix} m + M & -\frac{1}{2}mL \\ -\frac{1}{2}mL & \frac{1}{3}mL^2 \end{bmatrix} \quad \text{and} \quad [V_e] = \begin{bmatrix} k & 0 \\ 0 & -\frac{1}{2}mgL \end{bmatrix} \]

Then for \((x_e, \theta_e) = (0, 0)\), we have

\[ \det(-\omega^2[T_e] + [V_e]) = 0 \]
or
\[
\det \begin{bmatrix}
k - (m + M)\omega^2 & -\frac{1}{2}mL\omega^2 \\
-\frac{1}{2}mL\omega^2 & \frac{1}{2}mgL - \frac{1}{3}mL^2\omega^2
\end{bmatrix} = 0
\]
which leads to
\[
(m + 4M)\omega^4 - 2(2kL + 3(m + M)g)\omega^2 + 6kg = 0
\]
as before, and \((x_e, \theta_e) = (0, \pi)\), we have
\[
\det(-\omega^2[T_e] + [V_e]) = 0
\]
or
\[
\det \begin{bmatrix}
k - (m + M)\omega^2 & \frac{1}{2}mL\omega^2 \\
\frac{1}{2}mL\omega^2 & \frac{1}{2}mgL - \frac{1}{3}mL^2\omega^2
\end{bmatrix} = 0
\]
which leads to
\[
(m + 4M)\omega^4 - 2(2kL - 3(m + M)g)\omega^2 - 6kg = 0
\]
as before.