The Distribution of Inertia

5.1 The Center-Of-Mass of a System of Particles

Consider a system $S$ of $N$ point particles of masses $m_1$, $m_2$, $m_3$, ..., $m_N$ located at the positions $p_1$, $p_2$, $p_3$, ..., $p_N$, respectively, as measured from some reference point $O$. The center-of-mass ($C$) of the system of these $N$ point particles, as measured from point $O$, is defined by the vector

$$r_C = \frac{1}{m} \sum_{i=1}^{N} m_i p_i$$  \hspace{1cm} (5.1a)

where $m$ is the total mass of the system defined by

$$m = \sum_{i=1}^{N} m_i.$$  \hspace{1cm} (5.1b)

By writing Equation (5.1a) as

$$r_C = \sum_{i=1}^{N} \left( \frac{m_i}{m} \right) p_i \quad \text{with} \quad \sum_{i=1}^{N} \left( \frac{m_i}{m} \right) = 1$$

...
and $0 \leq m_i/m \leq 1$, we see that the center-of-mass vector can be viewed as a "weighted" statistical average of the positions $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \ldots, \mathbf{p}_N$, with $m_1/m$, $m_2/m$, $m_3/m$, $\ldots$, $m_N/m$ as the "statistical weight factors" for particles 1, 2, 3, $\ldots$, $N$, respectively. If we differentiate (as measured in A) Equation (5.1a) with respect to time once and twice, we get

$$\dot{\mathbf{r}}_C = \frac{1}{m} \sum_{i=1}^{N} m_i \dot{\mathbf{p}}_i \quad \text{and} \quad \ddot{\mathbf{r}}_C = \frac{1}{m} \sum_{i=1}^{N} m_i \ddot{a}_i \quad (5.2)$$

as the velocity and acceleration of the system’s center-of-mass, as measured from the reference frame A.

If the above system is continuous like a rigid body (instead of a discrete set of $N$ particles) then we write Equations (5.1a) and (5.1b) as

$$\mathbf{r}_C = \frac{1}{m} \int_{\text{system}} \mathbf{p} \, dm \quad (5.3a)$$

where $\mathbf{p}$ is some generic point in the system, as measured from point O, and $m$ is the total mass of the system defined by

$$m = \int_{\text{system}} dm, \quad (5.3b)$$

where:

- $dm = \lambda(\mathbf{p}) dl$ ($\lambda$ being a linear-mass density and $dl$ being a linear element) for a one-dimensional body,
- $dm = \sigma(\mathbf{p}) dA$ ($\sigma$ being an area-mass density and $dA$ being an area element) for a two-dimensional body, or
- $dm = \rho(\mathbf{p}) dV$ ($\rho$ being a volume-mass density and $dV$ being a volume element) for a three-dimensional body.

In general, when the system of discrete particles is replaced by a continuous body, we simply set

$$\mathbf{p}_i \to \mathbf{p} \quad \text{and} \quad \sum_{i=1}^{N} m_i \to \int_{\text{system}} dm.$$
Example #1: The Center of Mass for a Homogeneous Sector having Radius $R$ and Subtending an Angle $\theta$

The figure below shown a homogeneous planar sector having radius $R$ and subtending an angle $\theta$.

![Plot of the planar sector subtending the angle $\theta$ (indicated) and having radius $R$](image)

To compute the center of mass for a homogeneous planar sector having radius $R$ and subtending an angle $\theta$, we compute (in polar coordinates)

$$r_C = \frac{1}{m} \int \int_A \sigma p dA = \frac{\sigma \int \int_A (x \hat{e}_x + y \hat{e}_y) dA}{\sigma \int \int_A dA} = \frac{\int \int_A (x \hat{e}_x + y \hat{e}_y) dA}{\frac{1}{2} \theta R^2}$$

with

$$m = \int \int_A \sigma dA = \frac{1}{2} \sigma \theta R^2.$$

Thus we have

$$r_C = \frac{2}{\theta R^2} \int \int_A p dA = \frac{2}{\theta R^2} \int \int_A (x \hat{e}_x + y \hat{e}_y) dA,$$

and so (if we place the vertex of the sector at the origin and one side of the sector along the $x$-axis, as shown in the figure above), then $x = r \cos(\theta)$ and $y = r \sin(\theta)$ and, using polar coordinates, we have

$$r_C = \frac{2}{\theta R^2} \int_0^\theta \int_0^R (r \cos(\theta) \hat{e}_x + r \sin(\theta) \hat{e}_y) r dr d\theta$$
\[ r_C = \frac{2R^3}{3} \left( \frac{\sin(\theta)}{\theta} \hat{e}_x + \frac{1 - \cos(\theta)}{\theta} \hat{e}_y \right) \]  

which places the center of mass at the point 
\[ \left( \frac{2R \sin(\theta)}{3\theta}, \frac{2R(1 - \cos(\theta))}{3\theta} \right) \]
as measured from the vertex of the sector. ■ Note that for \( \theta = \pi/2 \), (a quarter circle) we get 
\[ r_C = \frac{4R}{3\pi} (\hat{e}_x + \hat{e}_y) . \]

**Example #2: The Center of Mass for a Planar Triangle**

Consider the triangle of sides \( a, b \) and \( c \), and angle \( \beta \), as shown in the figure below.

We may calculate the center of mass directly by using
\[ x_C = \frac{1}{m} \int_A x dm \quad \text{and} \quad y_C = \int_A y dm \]
with
\[ dm = \sigma dA = \sigma dydx \quad \text{and} \quad p = \hat{x}e_x + y\hat{e}_y. \]

This leads to
\[
x_C = \frac{1}{m} \sigma \int_0^{a\cos(\beta)} \int_0^{x\tan(\beta)} \quad \text{and} \quad p = \hat{x}e_x + y\hat{e}_y.
\]

which reduces to
\[
x_C = \frac{1}{3m} \sigma a^3 \cos^2(\beta) \sin(\beta) - \frac{1}{3m} \sigma a^3 \cos^2(\beta) \sin(\beta)
+ \frac{1}{6m} \sigma a^2 \sin(\beta) c \cos(\beta) + \frac{1}{6m} \sigma a \sin(\beta) c^2
\]
or
\[
x_C = \frac{1}{6m} \sigma a^2 \sin(\beta) c \cos(\beta) + \frac{1}{6m} \sigma ac^2 \sin(\beta).
\]

But
\[
\sigma = \frac{m}{A} \quad \text{with} \quad A = \frac{1}{2} ac \sin(\beta)
\]
and so
\[
\sigma = \frac{m}{A} = \frac{m}{2ac \sin(\beta)} = \frac{2m}{ac \sin(\beta)}.
\]

Then
\[
x_C = \frac{2m}{ac \sin(\beta)} \left( \frac{1}{6m} a^2 c \sin(\beta) \cos(\beta) + \frac{1}{6m} ac^2 \sin(\beta) \right)
\]
or
\[
x_C = \frac{1}{3} (a \cos(\beta) + c).
\]

We also have
\[
y_C = \frac{1}{m} \sigma \int_0^{a\cos(\beta)} \int_0^{x\tan(\beta)} \quad \text{and} \quad p = \hat{x}e_x + y\hat{e}_y.
\]

which reduces to
\[
y_C = \frac{1}{6} \sigma a^2 c \sin^2(\beta) = \frac{2m}{ac \sin(\beta)} \left( \frac{1}{6} a^2 c \sin^2(\beta) \right)
\]
or
\[
y_C = \frac{1}{3} a \sin(\beta).
\]
Thus we find that
\[ \mathbf{r}_C = \frac{1}{3} (a \cos(\beta) + c) \hat{e}_x + \frac{1}{3} a \sin(\beta) \hat{e}_y. \]

By writing this as
\[ \mathbf{r}_C = \frac{1}{3} (a \cos(\beta) \hat{e}_x + a \sin(\beta) \hat{e}_y) + \frac{1}{3} c \hat{e}_x \]
we see that
\[ \mathbf{r}_C = \frac{1}{3} (\mathbf{p}_L + \mathbf{p}_B) \quad (5.5a) \]

where
\[ \mathbf{p}_L = a \cos(\beta) \hat{e}_x + a \sin(\beta) \hat{e}_y \quad \text{and} \quad \mathbf{p}_B = c \hat{e}_x \]
are the left and bottom sides of the triangle, respectively, that are adjacent to the point O. Therefore, the position of the center of mass of a triangle, as measured from one of its vertices is simply one-third the sum of the two sides adjacent to this vertex.

As a special case we see that the center of mass of a right triangle, as measured from the right angle is one-third along one leg plus one-third along the other leg as shown in the figure below.

Incidently, it is useful to recall that the area of a triangle having side lengths \( a, b, \) and \( c \) is given by
\[ A = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{where} \quad s = \frac{a + b + c}{2}. \quad (5.6a) \]
Example #3: The Center of Mass for a Planar Triangle - Generalized

Consider the triangle that has been oriented so that its vertices are at $(x_1, 0)$, $(x_2, y_2)$ and $(x_3, 0)$, as shown in the figure below.

A Triangle with three vertices $(x_1, 0)$ (lower left vertex), $(x_2, y_2)$ (upper vertex) and $(x_3, 0)$ (lower right vertex)

We may calculate the center of mass directly by using

\[
x_C = \frac{1}{m} \int_A x \, dm \quad \text{and} \quad y_C = \int_A y \, dm
\]

with

\[
dm = \sigma dA = \sigma dy \, dx \quad \text{and} \quad \mathbf{p} = x\mathbf{e}_x + y\mathbf{e}_y.
\]

This leads to

\[
x_C = \frac{1}{m} \sigma \int_{x_1}^{x_2} \int_0^{y_2/(x_2-x_1)} x \, dy \, dx + \frac{1}{m} \sigma \int_{x_2}^{x_3} \int_0^{y_2/(x_3-x_2)} x \, dy \, dx
\]

which reduces to

\[
x_C = \frac{1}{6m} \sigma (2x_2^2 - x_1^2 - x_2x_1) y_2 + \frac{1}{6m} \sigma (x_3^2 + x_3x_2 - 2x_2^2) y_2
\]

or

\[
x_C = \frac{\sigma y_2(x_3 - x_1)(x_1 + x_2 + x_3)}{6m}
\]
But

\[ \sigma = \frac{m}{A} \quad \text{and} \quad A = \frac{1}{2}y_2(x_3 - x_1) \]

and so we get

\[ x_C = \frac{y_2(x_3 - x_1)(x_1 + x_2 + x_3)}{\frac{1}{2}y_2(x_3 - x_1)6} = \frac{x_1 + x_2 + x_3}{3}. \]

We also have

\[ y_C = \frac{1}{m} \sigma \int_{x_1}^{x_2} \int_{0}^{y_2(x_2 - x_1)/(x_2 - x_1)} ydydx + \frac{1}{m} \sigma \int_{x_2}^{x_3} \int_{0}^{y_2(x_3 - x)/(x_3 - x_2)} ydydx \]

which reduces to

\[ y_C = \frac{1}{6m} \sigma y_2^2(x_2 - x_1) + \frac{1}{6m} \sigma y_2^2(x_3 - x_2) \]

or

\[ y_C = \frac{\sigma y_2^2(x_3 - x_1)}{6m} = \frac{y_2^2(x_3 - x_1)}{\frac{1}{2}y_2(x_3 - x_1)6} = \frac{y_2}{3}. \]

Thus we find that

\[ \mathbf{r}_C = \frac{1}{3}(x_1 + x_2 + x_3)\mathbf{e}_x + \frac{1}{3}y_2\mathbf{e}_y \]

By writing this as

\[ \mathbf{r}_C = \frac{1}{3}(x_1\mathbf{e}_x + 0\mathbf{e}_y) + \frac{1}{3}(x_2\mathbf{e}_x + y_2\mathbf{e}_y) + \frac{1}{3}(x_3\mathbf{e}_x + 0\mathbf{e}_y) \]

we see that

\[ \mathbf{r}_C = \frac{1}{3}(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) \quad (5.5b) \]

where \( \mathbf{p}_1, \mathbf{p}_2 \) and \( \mathbf{p}_3 \) are the positions of the three vertices of the triangle as measured by point O. Therefore, the position of the center of mass of a triangle, as measured from some point O is simply the average of the positions of its vertices as measured from O. ■

Note that this is a very general result so that the center of mass for a uniform triangle having vertices located at (1, 2, 3), (8, 5, 3) and (2, 4, 8) is at

\[ \left( \frac{1 + 8 + 5}{3}, \frac{2 + 5 + 4}{3}, \frac{3 + 3 + 8}{3} \right) = \left( \frac{14}{3}, \frac{11}{3}, \frac{14}{3} \right) \]
Incidently, it is useful to recall that the area of a triangle having vertices \((x_1, y_1)\), \((x_2, y_2)\) and \((x_3, y_3)\) is given by

\[
A = \frac{1}{2} \left| \det \begin{bmatrix} x_1 & 1 & 1 \\ x_2 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \right|.
\]

(5.6b)

More generally, the area of the triangle having vertices \((x_1, y_1, z_2)\), \((x_2, y_2, z_2)\) and \((x_3, y_3, z_3)\) is given by

\[
A = \frac{1}{2} |(p_2 - p_1) \times (p_3 - p_1)|
\]

(5.6c)

where

\[
p_1 = x_1 \hat{e}_x + y_1 \hat{e}_y + z_1 \hat{e}_z,
\]

\[
p_2 = x_2 \hat{e}_x + y_2 \hat{e}_y + z_2 \hat{e}_z
\]

and

\[
p_3 = x_3 \hat{e}_x + y_3 \hat{e}_y + z_3 \hat{e}_z.
\]

Center-Of-Mass of a System of \(n\) Objects

It is easy to show that we may extend the above definition to a system of \(n\) objects (instead of just points) as follows. Suppose that there is a system of \(n\) objects of masses \(m_1, m_2, m_3, \ldots, m_n\) that have their centers-of-mass located at the positions \(r_{1C}, r_{2C}, r_{3C}, \ldots, r_{nC}\), respectively, as measured from some reference point \(O\). The center-of-mass (C) of this system of \(n\) objects can be shown (and the student should do this) to be given by the vector

\[
r_C = \frac{1}{m} \sum_{i=1}^{n} m_i r_{iC}
\]

(5.7a)

where

\[
m = \sum_{i=1}^{n} m_i,
\]

(5.7b)

is the total mass of the \(n\)-object system.
Example #4: A Grandfather’s Clock Pendulum

The figure below shows a grandfather’s clock pendulum, consisting of a uniform rod of length $L$ and mass $M$ with a solid disk of radius $R$ and mass $m$ attached to its bottom end. We want to determine the position of the pendulum’s center of mass, as measured from the top of the pendulum.

To solve this we treat the pendulum as two objects, a uniform rod of length $L$ and mass $M$, which has its center of mass a position $L/2$ below the top of the pendulum, and a uniform disk of radius $R$ and mass $m$, which has its center of mass a position $L + R$ below the top of the pendulum. Using Equation (5.7a), we then may say that the position of the pendulum’s center of mass, from the top of the pendulum is given by

$$ r_{\text{pendulum}, C} = \frac{m_{\text{rod}} r_{\text{rod}, C} + m_{\text{disk}} r_{\text{disk}, C}}{m_{\text{rod}} + m_{\text{disk}}} = \frac{M(L/2) + m(L + R)}{M + m}. $$

Example #5: The Center of Mass for a Triangle - Revisited

Consider the triangle of sides $a$, $b$ and $c$ as shown in the figure below.
A Triangle with sides $a$ (left side), $b$ (right side) and $c$ (bottom side)
The angle $\beta$ is the angle on the left vertex at O

We may calculate the center of mass thinking of this as two right triangles, the one on the left of sides $a \cos(\beta)$ and $a \sin(\beta)$ and the one on the right of sides $c - a \cos(\beta)$ and $a \sin(\beta)$. The position of the center of mass of each of these (as measured from O) are

$\left(a \cos(\beta) - \frac{1}{3}a \cos(\beta), \frac{1}{3}a \sin(\beta)\right) = \left(\frac{2}{3}a \cos(\beta), \frac{1}{3}a \sin(\beta)\right)$

and

$\left(a \cos(\beta) + \frac{1}{3}(c - a \cos(\beta)), \frac{1}{3}a \sin(\beta)\right) = \left(\frac{1}{3}c + \frac{2}{3}a \cos(\beta), \frac{1}{3}a \sin(\beta)\right)$
and these are indicated in the figure below.

![A Triangle with sides $a$ (left side), $b$ (right side) and $c$ (bottom side). The angle $\beta$ is the angle on the left vertex at O.]

The area of the left triangle is

$$A_{\text{left}} = \frac{1}{2}a \cos(\beta) a \sin(\beta) = \frac{1}{2}a^2 \sin(\beta) \cos(\beta)$$

and the area of the right triangle is

$$A_{\text{right}} = \frac{1}{2}(c - a \cos(\beta)) a \sin(\beta) = \frac{1}{2}a \sin(\beta)(c - a \cos(\beta))$$

and assuming uniform density, we have

$$\mathbf{r}_C = \frac{A_{\text{left}} \mathbf{r}_{\text{left},C} + A_{\text{right}} \mathbf{r}_{\text{right},C}}{A_{\text{left}} + A_{\text{right}}}$$

with

$$A_{\text{left}} + A_{\text{right}} = \frac{1}{2}a^2 \sin(\beta) \cos(\beta) + \frac{1}{2}a \sin(\beta)(c - a \cos(\beta)) = \frac{1}{2}ac \sin(\beta)$$

and so

$$\mathbf{r}_C = \frac{A_{\text{left}} \mathbf{r}_{\text{left},C} + A_{\text{right}} \mathbf{r}_{\text{right},C}}{A_{\text{left}} + A_{\text{right}}}$$

$$= \frac{\frac{1}{2}a^2 \sin(\beta) \cos(\beta)}{\frac{1}{2}ac \sin(\beta)} \left( \frac{2}{3}a \cos(\beta) \mathbf{e}_x + \frac{1}{3}a \sin(\beta) \mathbf{e}_y \right)$$
\[
\begin{align*}
&= a \cos(\beta) \left( \frac{2}{3} c + \frac{2}{3} a \cos(\beta) \right) \hat{e}_x + \frac{1}{3} a \sin(\beta) \hat{e}_y \\
&\quad + \left( 1 - \frac{a \cos(\beta)}{c} \right) \left( \left( \frac{1}{3} c + \frac{2}{3} a \cos(\beta) \right) \hat{e}_x + \frac{1}{3} a \sin(\beta) \hat{e}_y \right)
\end{align*}
\]

or
\[
r_C = \frac{1}{3} \left( c + a \cos(\beta) \right) \hat{e}_x + \frac{1}{3} a \sin(\beta) \hat{e}_y
\]

which is the result obtained earlier. ■

**Example #6: The Center of Mass for a Five-Sided Figure**

Determine the center of mass for the five-sided figure shown below assuming uniform mass density.

The vertices are located at the points: \((0, 0), (40, 0), (40, 20), (20, 60),\) and \((0, 80).\)
To solve this, we may divide the region into the three triangles shown below.

The center of mass and area of each of these triangles can be computed using Equations (5.5b and 5.6b) yielding

\[ A_{\text{left}} = \frac{1}{2} \left| \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 20 & 0 \\ 0 & 60 & 80 \end{bmatrix} \right| = 800 \]

and

\[ \mathbf{r}_{\text{left,C}} = \left( \frac{0 + 20 + 0}{3} \right) \mathbf{\hat{e}}_x + \left( \frac{0 + 60 + 80}{3} \right) \mathbf{\hat{e}}_y = \frac{20}{3} \mathbf{\hat{e}}_x + \frac{140}{3} \mathbf{\hat{e}}_y \]

and

\[ A_{\text{middle}} = \frac{1}{2} \left| \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 20 & 40 \\ 0 & 60 & 0 \end{bmatrix} \right| = 1200 \]

and

\[ \mathbf{r}_{\text{middle,C}} = \left( \frac{0 + 20 + 40}{3} \right) \mathbf{\hat{e}}_x + \left( \frac{0 + 60 + 0}{3} \right) \mathbf{\hat{e}}_y = \frac{20}{3} \mathbf{\hat{e}}_x + 20 \mathbf{\hat{e}}_y \]

and

\[ A_{\text{right}} = \frac{1}{2} \left| \det \begin{bmatrix} 1 & 1 & 1 \\ 20 & 40 & 40 \\ 60 & 20 & 0 \end{bmatrix} \right| = 200 \]

and

\[ \mathbf{r}_{\text{right,C}} = \left( \frac{20 + 40 + 40}{3} \right) \mathbf{\hat{e}}_x + \left( \frac{60 + 20 + 0}{3} \right) \mathbf{\hat{e}}_y = \frac{100}{3} \mathbf{\hat{e}}_x + \frac{80}{3} \mathbf{\hat{e}}_y. \]
Then
\[ \mathbf{r}_C = \frac{A_{\text{left}} \mathbf{r}_{\text{left},C} + A_{\text{middle}} \mathbf{r}_{\text{middle},C} + A_{\text{right}} \mathbf{r}_{\text{right},C}}{A_{\text{left}} + A_{\text{middle}} + A_{\text{right}}} \]
resulting in
\[ \mathbf{r}_C = \frac{4}{11} \left( \frac{20}{3} \hat{e}_x + \frac{140}{3} \hat{e}_y \right) + \frac{6}{11} \left( 20 \hat{e}_x + 20 \hat{e}_y \right) + \frac{1}{11} \left( \frac{100}{3} \hat{e}_x + \frac{80}{3} \hat{e}_y \right) \]
and reducing to
\[ \mathbf{r}_C = \frac{180}{11} \hat{e}_x + \frac{1000}{33} \hat{e}_y. \]
Note that the end result is not just the average of the vertices like in a triangle since this average gives
\[ \mathbf{r}_{\text{average}} = \left( \frac{0 + 40 + 40 + 20 + 0}{5} \right) \hat{e}_x + \left( \frac{0 + 0 + 20 + 60 + 80}{5} \right) \hat{e}_y = 20 \hat{e}_x + 32 \hat{e}_y \]
which is not \( \mathbf{r}_C \). The vertex averaging rule applies only to triangles! \[ \blacksquare \]

**Mass Unbalance**

Suppose next that we have our system of \( N \) particles with positions \( \mathbf{p}_i \) as measured from some point \( O \) and suppose that \( \mathbf{p} \) is the position of some point \( P \), as measured from \( O \). The mass unbalance of the system, relative to point \( P \) and as measured from \( O \), is defined as
\[
\mathbf{u}_P = \sum_{i=1}^{N} m_i (\mathbf{p}_i - \mathbf{p}). \tag{5.8a}
\]
Using the center of mass concept this reduces to
\[
\mathbf{u}_P = \sum_{i=1}^{N} m_i \mathbf{p}_i - \sum_{i=1}^{N} m_i \mathbf{p} = m \left( \frac{1}{m} \sum_{i=1}^{N} m_i \mathbf{p}_i \right) - \left( \sum_{i=1}^{N} m_i \right) \mathbf{p} = m \mathbf{r}_C - m \mathbf{p}
\]
or simply
\[
\mathbf{u}_P = m (\mathbf{r}_C - \mathbf{p}). \tag{5.8b}
\]
If $P$ is the center of mass $C$ of the system and if $r_i$ is the position of the $i$th particle as measured from the system's center of mass ($C$), then by simple vector addition, we have

$$p_i = r_C + r_i.$$ 

and we see that

$$\sum_{i=1}^{N} m_i r_i = \sum_{i=1}^{N} m_i (p_i - r_C) = u_C = m(r_C - r_C) = 0 \quad (5.9)$$

showing that the mass unbalance of the system, relative to its center of mass point $C$ and as measured from $O$ is zero. This also says that the position of the system’s center-of-mass, as measured from the system’s center-of-mass is $0$, which is reasonable.

If we differentiate Equation (5.9) with respect to time once and twice, we get the equations

$$\sum_{i=1}^{N} m_i \dot{r}_i = 0 \quad \text{and} \quad \sum_{i=1}^{N} m_i \ddot{r}_i = 0$$

where $\dot{r}_i$ and $\ddot{r}_i$ are the velocity and acceleration, respectively, of the $i$th particle as measured from the system’s center-of-mass $C$.

### 5.2 Inertia Vectors and Inertia Scalars

Let $S$ be a system of $N$ particles of masses $m_1, m_2, m_3, \ldots, m_N$, with position vectors $p_1, p_2, p_3, \ldots, p_N$, as measured from point $O$, and let $\hat{n}_a$ be any unit vector. Then a vector $\mathbf{I}_a$, called the inertial vector of $S$ relative to $O$ for $\hat{n}_a$, which we shall simply denote by $\mathbf{I}_a$, is defined by

$$\mathbf{I}_a = \sum_{i=1}^{N} m_i p_i \times (\hat{n}_a \times p_i) \quad (5.10a)$$

For a continuous system, we replace Equation (5.10a) with

$$\mathbf{I}_a = \int_{\text{system}} \mathbf{p} \times (\hat{n}_a \times \mathbf{p}) dm. \quad (5.10b)$$

where $\mathbf{r}$ is some generic point in the system, as measured from point $O$. 

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The scalar \( O_{ab}^S \) (or simply \( I_{ab} \)) defined by

\[
I_{ab} = \mathbf{I}_a \cdot \hat{n}_b.
\]  

(5.11a)
is called the inertia scalar of \( S \) relative to point \( O \) for \( \hat{n}_a \) and \( \hat{n}_b \), where \( \hat{n}_a \) and \( \hat{n}_b \), are any unit vectors. Putting Equation (5.10a) into Equation (5.11a), we get

\[
I_{ab} = \sum_{i=1}^{N} m_i (\mathbf{p}_i \times (\hat{n}_a \times \mathbf{p}_i)) \cdot \hat{n}_b
\]

or simply

\[
I_{ab} = \sum_{i=1}^{N} m_i (\hat{n}_a \times \mathbf{p}_i) \cdot (\hat{n}_b \times \mathbf{p}_i)
\]  

(5.11b)

which shows that

\[
I_{ab} = I_{ba}.
\]  

(5.11c)

Using the vector identity

\[
(\hat{n}_a \times \mathbf{p}_i) \cdot (\hat{n}_b \times \mathbf{p}_i) = \det \begin{bmatrix}
\hat{n}_a \cdot \hat{n}_b & \hat{n}_a \cdot \mathbf{p}_i \\
\mathbf{p}_i \cdot \hat{n}_b & \mathbf{p}_i \cdot \mathbf{p}_i
\end{bmatrix},
\]

Equation (5.11b) can be written as

\[
I_{ab} = \sum_{i=1}^{N} m_i ((\hat{n}_a \cdot \hat{n}_b) p_i^2 - (\mathbf{p}_i \cdot \hat{n}_a) (\mathbf{p}_i \cdot \hat{n}_b))
\]  

(5.11d)

with \( p_i^2 = \mathbf{p}_i \cdot \mathbf{p}_i \).

When \( \hat{n}_a \neq \hat{n}_b \), \( I_{ab} \) is called the product of inertia of \( S \) relative to point \( O \) for the unit vectors \( \hat{n}_a \) and \( \hat{n}_b \). When \( \hat{n}_a = \hat{n}_b \), \( I_{aa} \) (which is denoted by \( I_a \)) is called
the moment of inertia of $S$ relative to point $O$ with respect to a line ($L_a$) passing through $O$ and parallel to $\hat{n}_a$. Note that in this case we have

$$I_a = \sum_{i=1}^{N} m_i (\hat{n}_a \times \mathbf{p}_i) \cdot (\hat{n}_a \times \mathbf{p}_i) = \sum_{i=1}^{N} m_i (\mathbf{p}_i^2 - (\hat{n}_a \cdot \mathbf{p}_i)^2) = \sum_{i=1}^{N} m_i |\hat{n}_a \times \mathbf{p}_i|^2$$

or

$$I_a = \sum_{i=1}^{N} m_i l_i^2$$

(5.12a)

where $l_i$ is simply the perpendicular distance from point $P_i$ to the line $L_a$. The radius of gyration of $S$ with respect to the line $L_a$ passing through $O$ and parallel to $\hat{n}_a$ is defined by $k_a$ via the equation

$$I_a \equiv mk_a^2$$

(5.12b)

with $m$ being the total mass of $S$. Putting in Equation (5.12a) we have

$$k_a \equiv \sqrt{\frac{I_a}{m}} = \sqrt{\frac{1}{m} \sum_{i=1}^{N} m_i l_i^2}$$

(5.12c)

as the radius of gyration of $S$ with respect to the line $L_a$ passing through $O$ and parallel to $\hat{n}_a$.

5.3 Mutually Perpendicular Unit Vectors

Knowledge of inertia vectors $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_3$ of a system $S$ relative to a point $O$ for a standard reference triad (SRT)

$$C = \{\hat{n}_1, \hat{n}_2, \hat{n}_3\}$$

can be used to determine $\mathbf{I}_a$ for any unit vector $\hat{n}_a$, since we may write

$$\hat{n}_a = a_1 \hat{n}_1 + a_2 \hat{n}_2 + a_3 \hat{n}_3 \quad \text{with} \quad a_i = \hat{n}_a \cdot \hat{n}_i$$

and putting this into Equation (5.10a) leads to

$$\mathbf{I}_a = \sum_{i=1}^{N} m_i \mathbf{p}_i \times ((a_1 \hat{n}_1 + a_2 \hat{n}_2 + a_3 \hat{n}_3) \times \mathbf{p}_i)$$
\[ I_a = a_1 I_1 + a_2 I_2 + a_3 I_3 = \sum_{j=1}^{3} a_j I_{jj} \]  

(5.13a)

where

\[ I_j = \sum_{i=1}^{N} m_i p_i \times (\hat{n}_j \times p_i) \]  

(5.13b)

for \( j = 1, 2, 3 \).

By writing

\[ \hat{n}_b = b_1 \hat{n}_1 + b_2 \hat{n}_2 + b_3 \hat{n}_3 \quad \text{with} \quad b_i = \hat{n}_b \cdot \hat{n}_i \]  

(5.14)

we see that

\[ I_{ab} = \sum_{j=1}^{3} a_j I_{jj} \cdot \hat{n}_b \]

\[ = \sum_{j=1}^{3} a_j \sum_{i=1}^{N} m_i p_i \times (\hat{n}_j \times p_i) \cdot \sum_{k=1}^{3} b_k \hat{n}_k \]

\[ = \sum_{j=1}^{3} \sum_{k=1}^{3} a_j \left( \sum_{i=1}^{N} m_i p_i \times (\hat{n}_j \times p_i) \cdot \hat{n}_k \right) b_k \]

or simply

\[ I_{ab} = \sum_{j=1}^{3} \sum_{k=1}^{3} a_j I_{jk} b_k \]  

(5.15a)

where

\[ I_{jk} \equiv \sum_{i=1}^{N} m_i p_i \times (\hat{n}_j \times p_i) \cdot \hat{n}_k = \sum_{i=1}^{N} m_i (\hat{n}_j \times p_i) \cdot (\hat{n}_k \times p_i) \]  

(5.15b)
Note that for \( \{ \mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3 \} \) a SRT, we have from Equation (5.9b)

\[
I_{jk} = \sum_{i=1}^{N} m_i \left( (\hat{\mathbf{n}}_j \cdot \hat{\mathbf{n}}_k)p_i^2 - (\mathbf{p}_i \cdot \hat{\mathbf{n}}_j)(\mathbf{p}_i \cdot \hat{\mathbf{n}}_k) \right)
\]

or

\[
I_{jk} = \sum_{i=1}^{N} m_i \left( \delta_{jk}p_i^2 - (\mathbf{p}_i \cdot \hat{\mathbf{n}}_j)(\mathbf{p}_i \cdot \hat{\mathbf{n}}_k) \right), \quad (5.15c)
\]

where \( \delta_{jk} \) equals 1 for \( j = k \) and zero when \( j \neq k \) and \( \mathbf{p}_i \cdot \hat{\mathbf{n}}_j \) and \( \mathbf{p}_i \cdot \hat{\mathbf{n}}_k \) are the scalar components of \( \mathbf{p}_i \) for the SRT \( \{ \mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3 \} \). For a continuous system

Equation (5.15c) reads

\[
I_{jk} = \int_{\text{system}} \left( \delta_{jk}p^2 - (\mathbf{p} \cdot \hat{\mathbf{n}}_j)(\mathbf{p} \cdot \hat{\mathbf{n}}_k) \right) dm \quad (5.15d)
\]

where \( \mathbf{p} \) is some generic point in the system as measured by \( O \), and \( p^2 = \mathbf{p} \cdot \mathbf{p} \).

### 5.4 Inertia Matrices and The Inertia Dyadic

The inertia scalars \( I_{jk} \) of a system \( S \) relative to a point \( O \) for a SRT

\[
C = \{ \hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_3 \}
\]

can be used to define a square matrix \([C \mathcal{I}_C]\), called the inertia matrix of \( S \) relative to \( O \) for \( C \) as follows.

\[
[C \mathcal{I}_C] = \begin{bmatrix}
I_{11} & I_{12} & I_{13} \\
I_{21} & I_{22} & I_{23} \\
I_{31} & I_{32} & I_{33}
\end{bmatrix}.
\]

(5.16a)

Using this, Equation (5.15a) simply becomes

\[
I_{ab} = C\{\mathbf{n}_a\}^T [C \mathcal{I}_C] C\{\mathbf{n}_a\} \quad (5.16b)
\]

where

\[
C\{\mathbf{n}_a\} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \text{and} \quad C\{\mathbf{n}_b\} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.
\]

(5.16c)

It is important to emphasize the fact that \([C \mathcal{I}_C]\) does depend on the SRT \( C \). For a different SRT, say

\[
D = \{ \mathbf{n}_1', \mathbf{n}_2', \mathbf{n}_3' \}
\]

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one gets a different inertia matrix $[^D I_D]$. Hence, when working with inertia matrices, one must keep in mind that each such matrix is associated with a specific SRT. By way of contrast, the use of dyadics enables one to deal with certain topics involving inertia vectors and/or inertia scalars in a _SRT-independent_ way.

**Example #7: A Point Mass $m$ located at $P(x,y,z)$**

Compute the inertial matrix $[^C I_C]$, relative to the rectangular SRT

$$C = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\} = \{\hat{e}_x, \hat{e}_y, \hat{e}_z\},$$

for a particle of mass $m$ located at the point $P(x,y,z)$, so that

$$p = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z.$$ 

To solve this, we simply have

$$I_{11} = m(y^2 + z^2) \quad , \quad I_{22} = m(x^2 + z^2) \quad , \quad I_{33} = m(x^2 + y^2)$$

along with

$$I_{12} = -mxy = I_{21} \quad , \quad I_{13} = -mxz = I_{31}$$

and

$$I_{23} = -myz = I_{32}$$

where $(x,y,z)$ is just the position of $P$ and measured from some origin $O$. Thus we have

$$[^C I_C] = m \begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{bmatrix} \quad (5.17)$$

as the inertial matrix about the point $O$ for a point mass $m$ located at $P(x,y,z)$, as measured by $O$. ■
Example #8: A Uniform Box of Lengths, $a$, $b$ and $c$ about one of its corners

Compute the inertial matrix $[^{C}I_{C}]$, relative to the rectangular SRT

$$C = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\} = \{\hat{e}_x, \hat{e}_y, \hat{e}_z\},$$

for a uniform box in the region $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$. To solve this we let $m$ be the mass of the box, so that

$$\rho = \frac{m}{abc} \quad \text{and} \quad p = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$$

is the mass density of the box. Then $dm = \rho dV = \rho dz dy dx$ and

$$I_{11} = \int_{0}^{a} \int_{0}^{b} \int_{0}^{c} (y^2 + z^2) \rho dz dy dx = \frac{1}{3} \rho abc (b^2 + c^2) = \frac{1}{3} m (b^2 + c^2)$$

and

$$I_{12} = -\int_{0}^{a} \int_{0}^{b} \int_{0}^{c} xy \rho dz dy dx = -\frac{1}{4} \rho a^2 b^2 c = -\frac{1}{4} m ab = I_{21}$$

and

$$I_{13} = -\int_{0}^{a} \int_{0}^{b} \int_{0}^{c} xz \rho dz dy dx = -\frac{1}{4} \rho a^2 bc^2 = -\frac{1}{4} m ac = I_{31}$$

and

$$I_{22} = \int_{0}^{a} \int_{0}^{b} \int_{0}^{c} (x^2 + z^2) \rho dz dy dx = \frac{1}{3} \rho abc (a^2 + c^2) = \frac{1}{3} m (a^2 + c^2)$$

and

$$I_{23} = -\int_{0}^{a} \int_{0}^{b} \int_{0}^{c} yz \rho dz dy dx = -\frac{1}{4} \rho ab^2 c^2 = -\frac{1}{4} m bc = I_{32}$$

and

$$I_{33} = \int_{0}^{a} \int_{0}^{b} \int_{0}^{c} (x^2 + y^2) \rho dz dy dx = \frac{1}{3} \rho abc (a^2 + b^2) = \frac{1}{3} m (a^2 + b^2).$$

Thus we have

$$[^{C}I_{C}] = \frac{1}{12} m \begin{bmatrix} 4(b^2 + c^2) & -3ab & -3ac \\ -3ab & 4(a^2 + c^2) & -3bc \\ -3ac & -3bc & 4(a^2 + b^2) \end{bmatrix} \quad (5.18)$$

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as the inertial matrix about the origin for a uniform box in the region $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$.

Example #9: One-Eight of a Sphere of radius $R$

Compute the inertial matrix $[C \mathcal{I}_c]$, relative to the rectangular SRT

$$C = \{ \hat{e}_1, \hat{e}_2, \hat{e}_3 \} = \{ \hat{e}_x, \hat{e}_y, \hat{e}_z \},$$

for a uniform one-eight of a sphere of mass $m$ and radius $R$ in the first octant. To solve this we let $m$ be the mass of the box, so that

$$\rho = \frac{m}{\frac{1}{8} \left( \frac{4}{3} \pi R^3 \right)} = \frac{6m}{\pi R^3}$$

is the mass density of the sphere. Then, using spherical coordinates

$$dm = \rho dV = \rho r^2 \sin(\varphi) dr d\varphi d\theta$$

and

$$\mathbf{p} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$$

with $x = r \cos(\theta) \sin(\varphi)$, $y = r \sin(\theta) \sin(\varphi)$, and $z = r \cos(\varphi)$, we get

$$I_{11} = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^R (y^2 + z^2) \rho r^2 \sin(\varphi) dr d\varphi d\theta$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^R (r^2 \sin^2(\theta) \sin^2(\varphi) + r^2 \cos^2(\varphi)) \rho r^2 \sin(\varphi) dr d\varphi d\theta$$

$$= \frac{1}{15} \rho R^5 \pi = \frac{1}{15} \frac{6m}{\pi R^3} R^5 \pi = \frac{2}{5} m R^2$$

and

$$I_{12} = -\int_0^{\pi/2} \int_0^{\pi/2} \int_0^R xy \rho r^2 \sin(\varphi) dr d\varphi d\theta$$

$$= -\int_0^{\pi/2} \int_0^{\pi/2} \int_0^R r \cos(\theta) \sin(\varphi) r \sin(\theta) \sin(\varphi) \rho r^2 \sin(\varphi) dr d\varphi d\theta$$

$$= -\frac{1}{15} \rho R^5 = -\frac{1}{15} \frac{6m}{\pi R^3} R^5 = -\frac{2m}{5\pi} R^2 = I_{21}$$

and

$$I_{13} = -\int_0^{\pi/2} \int_0^{\pi/2} \int_0^R xz \rho r^2 \sin(\varphi) dr d\varphi d\theta$$
\[ I_{22} = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^R (x^2 + z^2) \rho r^2 \sin(\varphi) dr d\varphi d\theta = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^R (r^2 \cos^2(\theta) \sin^2(\varphi) + r^2 \cos^2(\varphi)) \rho r^2 \sin(\varphi) dr d\varphi d\theta = \frac{1}{15} \rho R^5 \pi = \frac{2}{5} m R^2 \]

and

\[ I_{23} = -\int_0^{\pi/2} \int_0^{\pi/2} \int_0^R y z \rho r^2 \sin(\varphi) dr d\varphi d\theta = -\int_0^{\pi/2} \int_0^{\pi/2} \int_0^R r \sin(\theta) \sin(\varphi) r \cos(\varphi) \rho r^2 \sin(\varphi) dr d\varphi d\theta = -\frac{1}{15} \rho R^5 \pi = -\frac{2m}{5\pi} R^2 = I_{32} \]

and

\[ I_{33} = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^R (x^2 + y^2) \rho r^2 \sin(\varphi) dr d\varphi d\theta = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^R (r^2 \cos^2(\theta) \sin^2(\varphi) + r^2 \sin^2(\theta) \sin^2(\varphi)) \rho r^2 \sin(\varphi) dr d\varphi d\theta = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^R r^2 \sin^2(\varphi) \rho r^2 \sin(\varphi) dr d\varphi d\theta = \frac{1}{15} \rho R^5 \pi = \frac{2m}{5\pi} R^2. \]

Thus we have

\[
[C \mathcal{I}_c] = \frac{2}{5\pi} m R^2 \begin{bmatrix}
\pi & -1 & -1 \\
-1 & \pi & -1 \\
-1 & -1 & \pi 
\end{bmatrix}.
\]

as the inertial matrix about the origin for a uniform one-eighth of a sphere of mass \( m \) in the first octant and having radius \( R \).
Example #10: Cylindrical Coordinates

Compute the inertial matrix $[^C I_C ]$, relative to the rectangular SRT

$C = \{ \hat{e}_1, \hat{e}_2, \hat{e}_3 \} = \{ \hat{x}, \hat{y}, \hat{z} \},$

for one-eight of a uniform cylinder of mass $m$, height $H$ and radius $R$ in the first octant. To solve this we let $m$ be the mass of the box, so that

$$\rho = \frac{m}{\frac{1}{4}(\pi R^2 H)} = \frac{4m}{\pi H R^2}$$

is the mass density of the cylinder. Then, using cylinder coordinates

$$dm = \rho dV = \rho rdrd\theta dz$$

and

$$p = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$$

with $x = r \cos(\theta)$, $y = r \sin(\theta)$, and $z = z$, we get

$$I_{11} = \int_0^H \int_0^{\pi/2} \int_0^R (y^2 + z^2) \rho r dr d\theta dz = \int_0^H \int_0^{\pi/2} \int_0^R (r^2 \sin^2(\theta) + z^2) \rho r dr d\theta dz$$

$$= \frac{1}{48} H \rho \pi R^2 (3R^2 + 4H^2) = \frac{1}{48} H \left( \frac{4m}{\pi H R^2} \right) \pi R^2 (3R^2 + 4H^2)$$

$$= \frac{1}{12} m (3R^2 + 4H^2)$$

and

$$I_{12} = -\int_0^H \int_0^{\pi/2} \int_0^R x y \rho r dr d\theta dz = -\int_0^H \int_0^{\pi/2} \int_0^R r \cos(\theta) r \sin(\theta) \rho r dr d\theta dz$$

$$= -\frac{1}{8} H R^4 \rho = -\frac{1}{8} H R^4 \left( \frac{4m}{\pi H R^2} \right) = -\frac{m R^2}{2\pi} = I_{21}$$

and

$$I_{13} = -\int_0^H \int_0^{\pi/2} \int_0^R x z \rho r dr d\theta dz = -\int_0^H \int_0^{\pi/2} \int_0^R r \cos(\theta) z \rho r dr d\theta dz$$

$$= -\frac{1}{6} \rho R^3 H^2 = -\frac{1}{6} \left( \frac{4m}{\pi H R^2} \right) R^3 H^2 = -\frac{2m}{3\pi} H R = I_{31}$$
and

\[ I_{22} = \int_0^H \int_0^{\pi/2} \int_0^R (x^2 + z^2) \rho r dr d\theta dz = \int_0^H \int_0^{\pi/2} \int_0^R (r^2 \cos^2(\theta) + z^2) \rho r dr d\theta dz \]

\[ = \frac{1}{48} H \rho \pi R^2 (3R^2 + 4H^2) = \frac{1}{48} H \left( \frac{4m}{\pi HR^2} \right) \pi R^2 (3R^2 + 4H^2) \]

\[ = \frac{1}{12} m (3R^2 + 4H^2) \]

and

\[ I_{23} = -\int_0^H \int_0^{\pi/2} \int_0^R yz \rho r dr d\theta dz = -\int_0^H \int_0^{\pi/2} \int_0^R r \sin(\theta) z \rho r dr d\theta dz \]

\[ = -\frac{1}{6} \rho R^3 H^2 = -\frac{1}{6} \left( \frac{4m}{\pi HR^2} \right) R^3 H^2 = -\frac{2m}{3\pi} HR = I_{32} \]

and

\[ I_{33} = \int_0^H \int_0^{\pi/2} \int_0^R (x^2 + y^2) \rho r dr d\theta dz = \int_0^H \int_0^{\pi/2} \int_0^R r^2 \rho r dr d\theta dz \]

\[ = \frac{1}{8} \rho R^4 \pi H = \frac{1}{8} \left( \frac{4m}{\pi HR^2} \right) R^4 \pi H = \frac{1}{2} mR^2 \]

Thus we have

\[
^C \mathcal{I}_C = \frac{1}{12\pi} m \begin{bmatrix}
(3R^2 + 4H^2)\pi & -6R^2 & -8HR \\
-6R^2 & (3R^2 + 4H^2)\pi & -8HR \\
-8HR & -8HR & 6\pi
\end{bmatrix}.
\]

as the inertial matrix about the origin for a uniform one-eighth of a cylinder of mass \( m \), height \( H \) and radius \( R \) in the first octant.

At this point, let us present a review of vectors and dyadics. The discussion of linear operators and dyadics begins in Section 5.7 so that you may skip to that section if you do not need to review vectors.

### 5.5 The Vector Space of Geometric Arrows - A Review

Let \( G_3 \) be the set of all geometric arrows in space. Each of these arrows have a length (called its magnitude) and a direction in space (called its direction),
including the "zero" arrow (0) which has zero length and (by definition) any
direction in space. The beginning of an arrow is called its tail and the end of the
arrow is called its tip. Note that an arrow in \( G_3 \) need not be fixed in space since
the arrow can be translated anywhere in space without changing its magnitude
and direction, and hence not changing the arrow itself. The magnitude of an
arrow \( x \) in \( G_3 \) is denoted by the symbol \( |x| \), and the direction of an arrow \( x \) in
\( G_3 \) is denoted by the symbol \( \hat{x} \). A direction opposite that of \( x \) is denoted by the
symbol \( -\hat{x} \). Note that the direction of an arrow \( x \) in \( G_3 \) (i.e., \( \hat{x} \)) is also considered
as an arrow in \( G_3 \) which has the same direction as that of \( x \), but having unit magnitude, so that \( |\hat{x}| \equiv 1 \).

We now define arrow addition by the tip-to-tail rule. This says if you want to
add the arrows \( x \) and \( y \) in \( G_3 \), then first you translate \( y \) until its tail is placed
on the tip of \( x \). This forms a two-arrow tip-to-tail chain. The sum of \( x \) and \( y \) is
defined as that arrow which connects the tail of \( x \) to the tip of \( y \), i.e., its connects
the beginning of the two-arrow tip-to-tail chain to the end of the two-arrow tip-
to-tail chain.

More generally, if you want to add the \( n \) arrows \( x_1, x_2, x_3, \ldots, x_n \) in \( G_3 \), then
first translate \( x_2 \) until its tail is placed on the tip of \( x_1 \), then you translate \( x_3 \)
until its tail is placed on the tip of \( x_2 \), then you translate \( x_4 \) until its tail is placed
on the tip of \( x_3 \), etc., until you translate \( x_n \) until its tail is placed on the tip of
\( x_{n-1} \). This forms an \( n \)-arrow tip-to-tail chain. The sum of \( x_1, x_2, x_3, \ldots, x_n \) in
\( G_3 \) is defined as that arrow which connects the tail of \( x_1 \) to the tip of \( x_n \), i.e., its connects
the beginning of the \( n \)-arrow tip-to-tail chain to the end of the \( n \)-arrow tip-to-tail chain.

Using the real numbers \( R \) as the field of scalars, we next define scalar multi-
lication by the rule which says when an arrow \( x \) is multiplied by a scalar \( \alpha \), the
result is an arrow whose length is multiplied by \( |\alpha| \) (i.e., \( |\alpha x| = |\alpha||x| \)) and the
direction of the result is the same as that of \( x \) when \( \alpha > 0 \), or opposite that of \( x \)
when \( \alpha < 0 \). That is,

\[
\alpha \hat{x} = \begin{cases} 
+\hat{x}, & \text{when } \alpha > 0 \\
-\hat{x}, & \text{when } \alpha < 0
\end{cases}
\]  

(5.19)

Note that for \( \alpha = 0 \), we define \( \alpha x = 0 \), the vector having zero length and any
direction.
It can easily be shown that $G_3$ with the above definitions of arrow addition and scalar multiplication forms a 3-dimensional real vector space.

If two vectors $\mathbf{a}$ and $\mathbf{b}$ in $G_3$ are drawn so that their tails coincide, then they will form two rays coming from a common point. The smallest angle between these two rays is called the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, and is denoted by $\theta$. Recall that the dot product between two vectors in $G_3$ is defined by

$$ \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta) \quad (5.20a) $$

and the cross product between two vectors in $G_3$ is defined by

$$ \mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin(\theta)\mathbf{n} \quad (5.20b) $$

where $\mathbf{n}$ is a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ and satisfying the right-hand rule where you place the fingers of your right hand along $\mathbf{a}$ and then curl these fingers through the angle $\theta$ until your fingers are aligned with $\mathbf{b}$ at which point your thumb will be pointing in the direction of $\mathbf{n}$.

5.6 Basis Sets And Coordinate Matrix Representations

Using what you already know about the vector space $G_3$, i.e., the dot product, cross product, triple scalar product, etc., you know that a set of three vectors $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ in $G_3$ is a basis set in $G_3$ if and only if the triple scalar product between $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ is not zero, i.e., if and only if

$$ \mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3) \neq 0. $$

We may write any $\mathbf{x}$ in $G_3$ as a unique linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, i.e.,

$$ \mathbf{x} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3. $$

The scalars $\alpha_1$, $\alpha_2$, $\alpha_3$ written as a $3 \times 1$ column array is called the coordinate matrix representation of $\mathbf{x}$ with respect to this basis set, and we write this as

$$ B\{\mathbf{x}\} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}. $$
Note that coordinate vectors satisfy the linear equations

\[ B\{x + y\} = B\{x\} + B\{y\} \quad \text{and} \quad B\{\alpha x\} = \alpha B\{x\} \quad (5.21) \]

for any vectors \( x \) and \( y \) in \( G_3 \) and any scalar \( \alpha \) in \( R \). Note that in these notes, column matrices will be denoted by curly brackets \( \{ \bullet \} \) while square matrices are denoted by square brackets \( [ \bullet ] \).

If \( E = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\} \) is a SRT so that \( ||\hat{e}_1|| = ||\hat{e}_2|| = ||\hat{e}_3|| = 1 \), and

\[
\hat{e}_1 = \hat{e}_2 \times \hat{e}_3 \quad , \quad \hat{e}_2 = \hat{e}_3 \times \hat{e}_1 \quad \& \quad \hat{e}_3 = \hat{e}_1 \times \hat{e}_2. \quad (5.22)
\]

and if

\[
x = \alpha_1 \hat{e}_1 + \alpha_2 \hat{e}_2 + \alpha_3 \hat{e}_3,
\]

we know that \( \alpha_i = x \cdot \hat{e}_i \), for \( i = 1, 2, 3 \) and hence

\[
E\{x\} = \begin{cases} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{cases} = \begin{cases} x \cdot \hat{e}_1 \\ x \cdot \hat{e}_2 \\ x \cdot \hat{e}_3 \end{cases}
\]

so that

\[
x = (x \cdot \hat{e}_1)\hat{e}_1 + (x \cdot \hat{e}_2)\hat{e}_2 + (x \cdot \hat{e}_3)\hat{e}_3 \quad (5.23)
\]

for each \( x \) in \( G_3 \). Finally, recall that if

\[
x = \alpha_1 \hat{e}_1 + \alpha_2 \hat{e}_2 + \alpha_3 \hat{e}_3 \quad \text{and} \quad y = \beta_1 \hat{e}_1 + \beta_2 \hat{e}_2 + \beta_3 \hat{e}_3
\]

so that

\[
E\{x\} = \begin{cases} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{cases} \quad \text{and} \quad E\{y\} = \begin{cases} \beta_1 \\ \beta_2 \\ \beta_3 \end{cases}
\]

then the dot product between \( x \) and \( y \) can be written simply as

\[
x \cdot y = \alpha_1 \beta_1 + \alpha_2 \beta_2 + \alpha_3 \beta_3 = \begin{cases} \alpha_1 & \alpha_2 & \alpha_3 \end{cases} \begin{cases} \beta_1 \\ \beta_2 \\ \beta_3 \end{cases}
\]

or

\[
x \cdot y = E\{x\}^T E\{y\}. \quad (5.24)
\]

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5.7 Linear Operators Defined On $G_3$

An operator $O : G_3 \to G_3$ defined on $G_3$ is a rule that takes an element of $G_3$ as input and returns a unique element of $G_3$ as output. If $x$ is an input and $y$ is the corresponding output, we write

$$O(x) = y.$$ 

An operator $L : G_3 \to G_3$ defined on $G_3$ is said to be linear if

$$L(x_1 + x_2) = L(x_1) + L(x_2) \quad \text{and} \quad L(\alpha x_1) = \alpha L(x_1)$$

for all $x_1$ and $x_2$ in $G_3$, and all $\alpha$ in $R$.

Example #11: The Cross Product

Let $a$ be some fixed vector in $G_3$, the operator $C : G_3 \to G_3$ defined by

$$C(x) = a \times x$$

is a linear operator on $G_3$ since

$$C(x_1 + x_2) = a \times (x_1 + x_2) = (a \times x_1) + (a \times x_2) = C(x_1) + C(x_2)$$

and

$$C(\alpha x_1) = a \times (\alpha x_1) = \alpha (a \times x_1) = \alpha C(x_1)$$

for all $x_1$ and $x_2$ in $G_3$, and all $\alpha$ in $R$. ■

Example #12

Let $a$ and $b$ be fixed vectors in $G_3$, the operator $L : G_3 \to G_3$ defined by

$$L(x) = (a \cdot x)b$$

is a linear operator on $G_3$ since

$$L(x_1 + x_2) = (a \cdot (x_1 + x_2))b = (a \cdot x_1)b + (a \cdot x_2)b = L(x_1) + L(x_2)$$

and

$$L(\alpha x_1) = (a \cdot (\alpha x_1))b = \alpha (a \cdot x_1)b = \alpha L(x_1)$$
for all \(x_1\) and \(x_2\) in \(G_3\), and all \(\alpha\) in \(R\). □

### 5.8 Matrix Representations Of Linear Operators

Suppose that \(L : G_3 \to G_3\) is a linear operator defined on \(G_3\) such that
\[
L(x) = y
\]
and suppose that
\[
E = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\}
\]
is a SRT in \(G_3\). Then we know that we may write
\[
x = \alpha_1 \hat{e}_1 + \alpha_2 \hat{e}_2 + \alpha_3 \hat{e}_3 \quad \text{and} \quad y = \beta_1 \hat{e}_1 + \beta_2 \hat{e}_2 + \beta_3 \hat{e}_3.
\]
so that
\[
E\{x\} = \begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{bmatrix} \quad \text{and} \quad E\{y\} = \begin{bmatrix}
\beta_1 \\
\beta_2 \\
\beta_3
\end{bmatrix}.
\]
This leads to
\[
L(\alpha_1 \hat{e}_1 + \alpha_2 \hat{e}_2 + \alpha_3 \hat{e}_3) = \beta_1 \hat{e}_1 + \beta_2 \hat{e}_2 + \beta_3 \hat{e}_3
\]
or
\[
\alpha_1 L(\hat{e}_1) + \alpha_2 L(\hat{e}_2) + \alpha_3 L(\hat{e}_3) = \beta_1 \hat{e}_1 + \beta_2 \hat{e}_2 + \beta_3 \hat{e}_3. \tag{5.25}
\]
But each of \(L(\hat{e}_1), L(\hat{e}_2)\) and \(L(\hat{e}_3)\) are in \(G_3\) and so we may write these as
\[
L(\hat{e}_1) = \gamma_{11} \hat{e}_1 + \gamma_{21} \hat{e}_2 + \gamma_{31} \hat{e}_3 , \quad L(\hat{e}_2) = \gamma_{12} \hat{e}_1 + \gamma_{22} \hat{e}_2 + \gamma_{32} \hat{e}_3
\]
and
\[
L(\hat{e}_3) = \gamma_{13} \hat{e}_1 + \gamma_{23} \hat{e}_2 + \gamma_{33} \hat{e}_3
\]
for scalars \(\gamma_{ij}\) \((i, j = 1, 2, 3)\) so that
\[
E\{L(\hat{e}_1)\} = \begin{bmatrix}
\gamma_{11} \\
\gamma_{21} \\
\gamma_{31}
\end{bmatrix} , \quad E\{L(\hat{e}_2)\} = \begin{bmatrix}
\gamma_{12} \\
\gamma_{22} \\
\gamma_{32}
\end{bmatrix} \quad \text{and} \quad E\{L(\hat{e}_3)\} = \begin{bmatrix}
\gamma_{13} \\
\gamma_{23} \\
\gamma_{33}
\end{bmatrix}.
\]
Putting these into Equation (5.25) leads to
\[
\beta_1 \hat{e}_1 + \beta_2 \hat{e}_2 + \beta_3 \hat{e}_3 = \alpha_1 (\gamma_{11} \hat{e}_1 + \gamma_{21} \hat{e}_2 + \gamma_{31} \hat{e}_3) + \alpha_2 (\gamma_{12} \hat{e}_1 + \gamma_{22} \hat{e}_2 + \gamma_{32} \hat{e}_3) + \alpha_3 (\gamma_{13} \hat{e}_1 + \gamma_{23} \hat{e}_2 + \gamma_{33} \hat{e}_3)
\]

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or
\[ \beta_1 \hat{e}_1 + \beta_2 \hat{e}_2 + \beta_3 \hat{e}_3 = (\alpha_1 \gamma_{11} + \alpha_2 \gamma_{12} + \alpha_3 \gamma_{13}) \hat{e}_1 + (\alpha_1 \gamma_{21} + \alpha_2 \gamma_{22} + \alpha_3 \gamma_{23}) \hat{e}_2 + (\alpha_1 \gamma_{31} + \alpha_2 \gamma_{32} + \alpha_3 \gamma_{33}) \hat{e}_3 \]

so that
\[ \beta_1 = \alpha_1 \gamma_{11} + \alpha_2 \gamma_{12} + \alpha_3 \gamma_{13} \]
\[ \beta_2 = \alpha_1 \gamma_{21} + \alpha_2 \gamma_{22} + \alpha_3 \gamma_{23} \]
\[ \beta_3 = \alpha_1 \gamma_{31} + \alpha_2 \gamma_{32} + \alpha_3 \gamma_{33} \]

which we may write as the matrix equation
\[
\begin{pmatrix}
\beta_1 \\
\beta_2 \\
\beta_3 \\
\end{pmatrix} =
\begin{bmatrix}
\gamma_{11} & \gamma_{12} & \gamma_{13} \\
\gamma_{21} & \gamma_{22} & \gamma_{23} \\
\gamma_{31} & \gamma_{32} & \gamma_{33} \\
\end{bmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\end{pmatrix}
\]

or
\[
E\{y\} = [^E \mathcal{L}_E]^E\{x\} \quad \text{with} \quad [^E \mathcal{L}_E] \equiv \begin{bmatrix}
\gamma_{11} & \gamma_{12} & \gamma_{13} \\
\gamma_{21} & \gamma_{22} & \gamma_{23} \\
\gamma_{31} & \gamma_{32} & \gamma_{33} \\
\end{bmatrix}.
\]

The above \(3 \times 3\) matrix \([^E \mathcal{L}_E]\) is called the matrix representation of the linear operator \(\mathcal{L}\) with respect to the SRT
\[ E = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\} \]

Note that the columns of this matrix are
\[
^E\{\mathcal{L}(\hat{e}_1)\} = \begin{bmatrix}
\gamma_{11} \\
\gamma_{21} \\
\gamma_{31} \\
\end{bmatrix}, \quad ^E\{\mathcal{L}(\hat{e}_2)\} = \begin{bmatrix}
\gamma_{12} \\
\gamma_{22} \\
\gamma_{32} \\
\end{bmatrix}, \quad \& \quad ^E\{\mathcal{L}(\hat{e}_3)\} = \begin{bmatrix}
\gamma_{13} \\
\gamma_{23} \\
\gamma_{33} \\
\end{bmatrix}
\]

and so we may write \([^E \mathcal{L}_E]\) as
\[
[^E \mathcal{L}_E] = [^E\{\mathcal{L}(\hat{e}_1)\}] [^E\{\mathcal{L}(\hat{e}_2)\}] [^E\{\mathcal{L}(\hat{e}_3)\}]
\]

and
\[
^E\{\mathcal{L}(x)\} = [^E \mathcal{L}_E]^E\{x\}.
\]
Example #13: The Cross Product

Let

\[ E = \{ \hat{e}_1, \hat{e}_2, \hat{e}_3 \} \]

is a SRT \( G_3 \), and let

\[ s = s_1 \hat{e}_1 + s_2 \hat{e}_2 + s_3 \hat{e}_3 \]

be some fixed vector in \( G_3 \). We have seen that the operator \( C : G_3 \to G_3 \) defined by

\[ C(\mathbf{x}) = \mathbf{s} \times \mathbf{x} \]

is a linear operator on \( G_3 \). To compute \([E C_E]\), we use

\[ C(\hat{e}_1) = s \times \hat{e}_1 \quad , \quad C(\hat{e}_2) = s \times \hat{e}_2 \quad , \quad C(\hat{e}_3) = s \times \hat{e}_3 \]

which leads to

\[
C(\hat{e}_1) = s \times \hat{e}_1 \\
= (s_1 \hat{e}_1 + s_2 \hat{e}_2 + s_3 \hat{e}_3) \times \hat{e}_1 \\
= s_1 (\hat{e}_1 \times \hat{e}_1) + s_2 (\hat{e}_2 \times \hat{e}_1) + s_3 (\hat{e}_3 \times \hat{e}_1) \\
= s_1 (0) + s_2 (-\hat{e}_3) + s_3 (\hat{e}_2) \\
= (0)\hat{e}_1 + (s_3)\hat{e}_2 + (s_2)\hat{e}_3
\]

and

\[
C(\hat{e}_2) = s \times \hat{e}_2 \\
= (s_1 \hat{e}_1 + s_2 \hat{e}_2 + s_3 \hat{e}_3) \times \hat{e}_2 \\
= s_1 (\hat{e}_1 \times \hat{e}_2) + s_2 (\hat{e}_2 \times \hat{e}_2) + s_3 (\hat{e}_3 \times \hat{e}_2) \\
= s_1 (\hat{e}_3) + s_2 (0) + s_3 (-\hat{e}_1) \\
= (s_3)\hat{e}_1 + (0)\hat{e}_2 + (s_1)\hat{e}_3
\]

and

\[
C(\hat{e}_3) = s \times \hat{e}_3 \\
= (s_1 \hat{e}_1 + s_2 \hat{e}_2 + s_3 \hat{e}_3) \times \hat{e}_3 \\
= s_1 (\hat{e}_1 \times \hat{e}_3) + s_2 (\hat{e}_2 \times \hat{e}_3) + s_3 (\hat{e}_3 \times \hat{e}_3) \\
= s_1 (-\hat{e}_2) + s_2 (\hat{e}_1) + s_3 (0) \\
= (s_2)\hat{e}_1 + (-s_1)\hat{e}_2 + (0)\hat{e}_3
\]
This leads to
\[
E\{C(\hat{e}_1)\} = \begin{cases} 
0 & \text{s}_3 \\
-s_2 & \text{s}_3 \\
s_2 & \text{s}_3 
\end{cases}, \quad E\{C(\hat{e}_2)\} = \begin{cases} 
-s_3 & \text{s}_1 \\
0 & \text{s}_1 \\
s_1 & \text{s}_1 
\end{cases}, \quad E\{C(\hat{e}_3)\} = \begin{cases} 
s_2 & \text{s}_1 \\
-s_1 & \text{s}_1 \\
0 & \text{s}_1 
\end{cases}
\]
so that
\[
[E_C] = \begin{bmatrix} 0 & -s_3 & s_2 \\
s_3 & 0 & -s_1 \\
-s_2 & s_1 & 0 
\end{bmatrix}
\] (5.29)
which agrees with results seen much earlier in the semester.

### 5.9 Dyads

Suppose that \(a\) and \(b\) are two vectors in \(G_3\) and suppose \(D\) is the linear operator such that
\[
D(x) = (b \cdot x)a = a(b \cdot x)
\]
and suppose that
\[
E = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\}
\]
is a SRT in \(G_3\) with
\[
a = \alpha_1\hat{e}_1 + \alpha_2\hat{e}_2 + \alpha_3\hat{e}_3 \quad \text{and} \quad b = \beta_1\hat{e}_1 + \beta_2\hat{e}_2 + \beta_3\hat{e}_3.
\]
Then
\[
D(\hat{e}_1) = a(b \cdot \hat{e}_1) = (\beta_1)a = \beta_1(\alpha_1\hat{e}_1 + \alpha_2\hat{e}_2 + \alpha_3\hat{e}_3)
\]
\[
= (\alpha_1\beta_1)\hat{e}_1 + (\alpha_2\beta_1)\hat{e}_2 + (\alpha_3\beta_1)\hat{e}_3
\]
and
\[
D(\hat{e}_2) = a(b \cdot \hat{e}_2) = (\beta_2)a = \beta_2(\alpha_1\hat{e}_1 + \alpha_2\hat{e}_2 + \alpha_3\hat{e}_3)
\]
\[
= (\alpha_1\beta_2)\hat{e}_1 + (\alpha_2\beta_2)\hat{e}_2 + (\alpha_3\beta_2)\hat{e}_3
\]
and
\[
D(\hat{e}_3) = a(b \cdot \hat{e}_3) = (\beta_3)a = \beta_3(\alpha_1\hat{e}_1 + \alpha_2\hat{e}_2 + \alpha_3\hat{e}_3)
\]
\[
= (\alpha_1\beta_3)\hat{e}_1 + (\alpha_2\beta_3)\hat{e}_2 + (\alpha_3\beta_3)\hat{e}_3
\]
so that
\[ E\{D(\mathbf{e}_1)\} = \begin{bmatrix} \alpha_1 \beta_1 \\ \alpha_2 \beta_1 \\ \alpha_3 \beta_1 \end{bmatrix} , \quad E\{D(\mathbf{e}_2)\} = \begin{bmatrix} \alpha_1 \beta_2 \\ \alpha_2 \beta_2 \\ \alpha_3 \beta_2 \end{bmatrix} , \quad E\{D(\mathbf{e}_3)\} = \begin{bmatrix} \alpha_1 \beta_3 \\ \alpha_2 \beta_3 \\ \alpha_3 \beta_3 \end{bmatrix} \]
and
\[ [E D_E] = \begin{bmatrix} \alpha_1 \beta_1 & \alpha_1 \beta_2 & \alpha_1 \beta_3 \\ \alpha_2 \beta_1 & \alpha_2 \beta_2 & \alpha_2 \beta_3 \\ \alpha_3 \beta_1 & \alpha_3 \beta_2 & \alpha_3 \beta_3 \end{bmatrix} \] (5.30a)
which we may write as
\[ [E D_E] = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \end{bmatrix} = [E\{a\}]^T [E\{b\}]^T \]
and hence
\[ E\{D(\mathbf{x})\} = E\{a(\mathbf{b} \cdot \mathbf{x})\} = [E D_E] E\{\mathbf{x}\} = [E\{a\}]^T [E\{b\}]^T E\{\mathbf{x}\}. \]

Note that we could have also obtained this very quickly using the result
\[ \mathbf{x} \cdot \mathbf{y} = [E\{\mathbf{x}\}]^T [E\{\mathbf{y}\}] \]
so that
\[ E\{D(\mathbf{x})\} = E\{a(\mathbf{b} \cdot \mathbf{x})\} = E\{a\}(\mathbf{b} \cdot \mathbf{x}) = E\{a\}^T [E\{b\}]^T E\{\mathbf{x}\}. \]

At this point we are motivated to define an operator, known as a dyad that is constructed from vectors \( \mathbf{a} \) and \( \mathbf{b} \) as
\[ D \equiv \mathbf{a} \mathbf{b} \]
such that
\[ D \cdot \mathbf{x} = (\mathbf{a} \mathbf{b}) \cdot \mathbf{x} = \mathbf{a}(\mathbf{b} \cdot \mathbf{x}). \] (5.31)

This can be remembered by starting with \((\mathbf{a} \mathbf{b}) \cdot \mathbf{x}\) and moving \( \mathbf{a} \) to the left outside the parenthesis and moving \( \cdot \mathbf{x} \) to the left inside the parenthesis. It is clear that the linear operator \( D : G_3 \rightarrow G_3 \) takes a vector \( \mathbf{x} \) as input and returns the vector \( \mathbf{a}(\mathbf{b} \cdot \mathbf{x}) \) as output, and its matrix representation with respect to a SRT
\[ E = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \]

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is given by

\[ [E^D_E] = [E^{(ab)}_E] = E^E \{a\}^E \{b\}^T \]  \hspace{1cm} (5.32)

and

\[ E^E \{(ab) \cdot x\} = [E^{(ab)}_E]^E \{x\} = E^E \{a\}^E \{b\}^T E \{x\}. \]

Since \([E^{(ab)}_E] \) is a square matrix, we may also consider the product

\[ E^E \{x\}^T [E^{(ab)}_E] E^E \{x\} = E^E \{x\}^T E^E \{a\} E^E \{b\}^T = (x \cdot a)^E \{b\}^T = E^E \{(x \cdot a)b\}^T. \]

We shall write this as

\[ x \cdot (ab) = (x \cdot a)b. \]  \hspace{1cm} (5.33)

This can be remembered by starting with \( x \cdot (ab) \) and moving \( b \) to the right outside the parenthesis and moving \( x \cdot \) to the right inside the parenthesis. Note that the combination \( x \cdot (ab) \cdot y \) is a scalar which can be written as

\[ x \cdot (ab) \cdot y = ((x \cdot a)b) \cdot y = (x \cdot a)(b \cdot y) \]  \hspace{1cm} (5.34)

or

\[ x \cdot (ab) \cdot y = (x \cdot a)(b \cdot y) = E^E \{x\}^T E^E \{a\} E^E \{b\}^T E \{y\} \]

where \( E \) is any SRT in \( G_3 \).

Two dyads \( D_1 \) and \( D_2 \) are said to be equal \((D_1 = D_2)\) if and only if

\[ D_1(x) = D_2(x) \]

for all \( x \) in \( G_3 \). Note that since

\( (ab) \cdot x = a(b \cdot x) \)

we may interchange \( a \) and \( b \) and write

\( (ba) \cdot x = b(a \cdot x) \)

and since \( a(b \cdot x) \neq b(a \cdot x) \), we see that, in general \( ab \neq ba \).

Note that although in general

\[ (ab) \cdot c = a(b \cdot c) \neq c \cdot (ab) = (c \cdot a)b \]  \hspace{1cm} (5.35a)
we do have
\[(\mathbf{a}\mathbf{a}) \cdot \mathbf{b} = \mathbf{a}(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{b} \cdot \mathbf{a})\mathbf{a} = \mathbf{b} \cdot (\mathbf{a}\mathbf{a}).\] (5.35b)

### 5.10 Dyadics

Suppose that \(\alpha_1\) and \(\alpha_2\) are any scalars in \(\mathbb{R}\) and \(\mathbf{D}_1 = \mathbf{a}\mathbf{b}\) and \(\mathbf{D}_2 = \mathbf{c}\mathbf{d}\) are two dyads constructed from the vectors \(\mathbf{a}, \mathbf{b}, \mathbf{c},\) and \(\mathbf{d}.\) We may construct a linear operator \(\mathbf{D} : \mathbb{G}_3 \rightarrow \mathbb{G}_3\) as
\[
\mathbf{D} = \alpha_1 \mathbf{D}_1 + \alpha_2 \mathbf{D}_2 = \alpha_1 \mathbf{a}\mathbf{b} + \alpha_2 \mathbf{c}\mathbf{d}.
\]
such that
\[
\mathbf{D}(\mathbf{x}) = (\alpha_1 \mathbf{a}\mathbf{b} + \alpha_2 \mathbf{c}\mathbf{d}) \cdot \mathbf{x} = \alpha_1((\mathbf{a}\mathbf{b}) \cdot \mathbf{x}) + \alpha_2((\mathbf{c}\mathbf{d}) \cdot \mathbf{x}) = \alpha_1\mathbf{a}(\mathbf{b} \cdot \mathbf{x}) + \alpha_2\mathbf{c}(\mathbf{d} \cdot \mathbf{x}).
\]
This is a linear combination of 2 dyads.

A linear combination of the \(n\) dyads: \(\mathbf{D}_1 = \mathbf{a}_1\mathbf{b}_1, \mathbf{D}_2 = \mathbf{a}_2\mathbf{b}_2, \mathbf{D}_3 = \mathbf{a}_3\mathbf{b}_3,\ldots,\mathbf{D}_n = \mathbf{a}_n\mathbf{b}_n,\) is written as
\[
\mathbf{D} = \sum_{k=1}^{n} \alpha_k \mathbf{D}_k = \sum_{k=1}^{n} \alpha_k \mathbf{a}_k \mathbf{b}_k
\]
and this is called a dyadic, and if \(E\) is a SRT in \(\mathbb{G}_3,\) then it is clear that
\[
[E\mathbf{D}] = \sum_{k=1}^{n} \alpha_k [E(\mathbf{D}_k)] = \sum_{k=1}^{n} \alpha_k [E(\mathbf{a}_k\mathbf{b}_k)] = \sum_{k=1}^{n} \alpha_k [E\mathbf{a}_k] [E\mathbf{b}_k]^T.
\]

**Example #14: The Triple Cross Product**

We know from vector algebra that the operator \(T : \mathbb{G}_3 \rightarrow \mathbb{G}_3\) defined by
\[
T(\mathbf{x}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{x}
\]
can be written as
\[
T(\mathbf{x}) = (\mathbf{a} \cdot \mathbf{x})\mathbf{b} - (\mathbf{b} \cdot \mathbf{x})\mathbf{a} = \mathbf{b} (\mathbf{a} \cdot \mathbf{x}) - \mathbf{a} (\mathbf{b} \cdot \mathbf{x})
\]

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which we may now write as
\[ T(x) = (a \times b) \times x = (ba) \cdot x - (ab) \cdot x = (ba - ab) \cdot x, \]
and so
\[ T = (a \times b) \times \equiv ba - ab \quad (5.36) \]
showing that \( T \) is a dyadic operator. 

**Example #15: Unit Dyadics**

We know that if
\[ E = \{ \hat{e}_1, \hat{e}_2, \hat{e}_3 \}. \]
is a SRT in \( G_3 \), then for any vector \( x \) in \( G_3 \), we may write
\[ x = (\hat{e}_1 \cdot x)\hat{e}_1 + (\hat{e}_2 \cdot x)\hat{e}_2 + (\hat{e}_3 \cdot x)\hat{e}_3 \]
Let us now rearrange this as follows.
\[ x = (\hat{e}_1 \cdot x)\hat{e}_1 + (\hat{e}_2 \cdot x)\hat{e}_2 + (\hat{e}_3 \cdot x)\hat{e}_3 \]
\[ = \hat{e}_1(\hat{e}_1 \cdot x) + \hat{e}_2(\hat{e}_2 \cdot x) + \hat{e}_3(\hat{e}_3 \cdot x) \]
\[ = (\hat{e}_1\hat{e}_1) \cdot x + (\hat{e}_2\hat{e}_2) \cdot x + (\hat{e}_3\hat{e}_3) \cdot x \]
\[ = (\hat{e}_1\hat{e}_1 + \hat{e}_2\hat{e}_2 + \hat{e}_3\hat{e}_3) \cdot x \]
which we can now write as
\[ x = \mathcal{U} \cdot x \quad \text{with} \quad \mathcal{U} \equiv \hat{e}_1\hat{e}_1 + \hat{e}_2\hat{e}_2 + \hat{e}_3\hat{e}_3. \quad (5.37) \]
The dyadic \( \mathcal{U} \) is called a *unit* dyadic. It is important to note that the form of \( \mathcal{U} \) in this equation is the same for any SRT, and
\[
\begin{bmatrix}
E\mathcal{U}_E
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]
5.11 The Vector Space Of All Dyadics Defined On $G_3$

From our above discussions, it is clear that if $D_1$ and $D_2$ are any two dyadics defined on $G_3$, then

$$D_1 + D_2$$ such that $$(D_1 + D_2)(x) = D_1(x) + D_2(x) \tag{5.38a}$$

is also a dyadic defined on $G_3$ and

$$\alpha D_1$$ such that $$(\alpha D_1)(x) = \alpha D_1(x) \tag{5.38b}$$

for any scalar $\alpha$ in $\mathbb{R}$ is also a dyadic defined on $G_3$. It is easy to show that the set of all possible dyadics defined on $G_3$ with the above operations of dyadic addition and scalar multiplication forms a vector space and we shall refer to this vector space as $G_9$.

We know that the dimension of the vector space $G_3$ is 3. Let us now show that the dimension of $G_9$ is $3^2 = 9$. Toward this end suppose that

$$E = \{ \hat{e}_1, \hat{e}_2, \hat{e}_3 \}.$$  

is a SRT in $G_3$. Then the dyad $ab$ for

$$a = \alpha_1 \hat{e}_1 + \alpha_2 \hat{e}_2 + \alpha_3 \hat{e}_3 \quad \text{and} \quad b = \beta_1 \hat{e}_1 + \beta_2 \hat{e}_2 + \beta_3 \hat{e}_3.$$  

has the matrix representation

$$[E(ab)_{ES}] = E\{ a \} E\{ b \}^T = \begin{bmatrix} \alpha_1 \beta_1 & \alpha_1 \beta_2 & \alpha_1 \beta_3 \\ \alpha_2 \beta_1 & \alpha_2 \beta_2 & \alpha_2 \beta_3 \\ \alpha_3 \beta_1 & \alpha_3 \beta_2 & \alpha_3 \beta_3 \end{bmatrix}, \tag{5.39}$$

which we may write as

$$E\{ a \} E\{ b \}^T = \alpha_1 \beta_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \alpha_1 \beta_2 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \alpha_1 \beta_3 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\quad + \alpha_2 \beta_1 \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \alpha_2 \beta_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \alpha_2 \beta_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\quad + \alpha_3 \beta_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \alpha_3 \beta_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \alpha_3 \beta_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$
or

\[
E\{a\} E\{b\}^T = \alpha_1 \beta_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^T + \alpha_1 \beta_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^T + \alpha_1 \beta_3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^T + \alpha_2 \beta_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^T + \alpha_2 \beta_2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^T + \alpha_2 \beta_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}^T + \alpha_3 \beta_1 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^T + \alpha_3 \beta_2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}^T + \alpha_3 \beta_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^T
\]

or simply

\[
E\{a\} E\{b\}^T = \alpha_1 \beta_1 E\{\hat{e}_1\} E\{\hat{e}_1\}^T + \alpha_1 \beta_2 E\{\hat{e}_1\} E\{\hat{e}_2\}^T + \alpha_1 \beta_3 E\{\hat{e}_1\} E\{\hat{e}_3\}^T + \alpha_2 \beta_1 E\{\hat{e}_2\} E\{\hat{e}_1\}^T + \alpha_2 \beta_2 E\{\hat{e}_2\} E\{\hat{e}_2\}^T + \alpha_2 \beta_3 E\{\hat{e}_2\} E\{\hat{e}_3\}^T + \alpha_3 \beta_1 E\{\hat{e}_3\} E\{\hat{e}_1\}^T + \alpha_3 \beta_2 E\{\hat{e}_3\} E\{\hat{e}_2\}^T + \alpha_3 \beta_3 E\{\hat{e}_3\} E\{\hat{e}_3\}^T
\]

Thus we may write

\[
[\{E(ab)\}] = E\{a\} E\{b\}^T = \sum_{i=1}^{3} \sum_{j=1}^{3} \alpha_i \beta_j \sum_{i=1}^{3} \sum_{j=1}^{3} \alpha_i \beta_j \hat{e}_i \hat{e}_j = \left( \sum_{i=1}^{3} \sum_{j=1}^{3} \alpha_i \beta_j \hat{e}_i \hat{e}_j \right)_{E}\]

for any dyad \(ab\). Since any dyadic can be written as a linear combinations of dyads \(ab\), we see then that any dyadic can also be written as

\[
Q = \sum_{i=1}^{3} \sum_{j=1}^{3} Q_{ij} \hat{e}_i \hat{e}_j, \quad (5.41)
\]

and so the set of 9 dyads

\[
E = \{\hat{e}_i \hat{e}_j \mid i = 1, 2, 3 \text{ and } j = 1, 2, 3\}
\]
span all of $G_9$. It is easy to show that this set is linearly independent and hence the above set of 9 dyads form a basis set for the vector space of all dyadics $G_9$ defined on $G_3$, and hence the dimension of $G_9$ is 9.

Thus we see that if 
\[
E = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\},
\]
is a SRT in $G_3$, then
\[
E = \{\hat{e}_i \hat{e}_j \mid i = 1, 2, 3 \text{ and } j = 1, 2, 3\}
\]
is a basis set for $G_9$. In addition we note that for any $Q$ in $G_9$, we have
\[
Q = \sum_{i=1}^{3} \sum_{j=1}^{3} Q_{ij} \hat{e}_i \hat{e}_j,
\]
and for any
\[
u = u_1 \hat{e}_1 + u_2 \hat{e}_2 + u_3 \hat{e}_3
\]
in $G_3$, we have
\[
Q \cdot \nu = \sum_{i=1}^{3} \sum_{j=1}^{3} Q_{ij}(\hat{e}_i \hat{e}_j) \cdot \nu = \sum_{i=1}^{3} \sum_{j=1}^{3} Q_{ij} \hat{e}_i (\hat{e}_j \cdot \nu) = \sum_{i=1}^{3} \sum_{j=1}^{3} Q_{ij} \hat{e}_i u_j
\]
or
\[
Q \cdot \nu = \sum_{i=1}^{3} \left( \sum_{j=1}^{3} Q_{ij} u_j \right) \hat{e}_i
\]
so that
\[
E\{Q \cdot \nu\} = \begin{bmatrix}
Q_{11} u_1 + Q_{12} u_2 + Q_{13} u_3 \\
Q_{21} u_1 + Q_{22} u_2 + Q_{23} u_3 \\
Q_{31} u_1 + Q_{32} u_2 + Q_{33} u_3
\end{bmatrix}
= \begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} \\
Q_{21} & Q_{22} & Q_{23} \\
Q_{31} & Q_{32} & Q_{33}
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix}
\]
or simply
\[
E\{Q \cdot \nu\} = [E Q_E]^E\{\nu\}
\]
so that
\[
[E Q_E] = \begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} \\
Q_{21} & Q_{22} & Q_{23} \\
Q_{31} & Q_{32} & Q_{33}
\end{bmatrix}
\]

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is the matrix representation of the dyadic $Q$ with respect to the basis set $E$ in $G_3$. Note also that for 

$$v = v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3$$

in $G_3$, we have

$$v \cdot Q = \sum_{i=1}^{3} \sum_{j=1}^{3} Q_{ij}v \cdot (\hat{e}_i\hat{e}_j) = \sum_{i=1}^{3} \sum_{j=1}^{3} Q_{ij}(v \cdot \hat{e}_i)\hat{e}_j = \sum_{i=1}^{3} \sum_{j=1}^{3} Q_{ij}v_i\hat{e}_j$$

or

$$v \cdot Q = \sum_{j=1}^{3} \left( \sum_{i=1}^{3} v_i Q_{ij} \right) \hat{e}_j$$

so that

$$E\{v \cdot Q\} = \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

or

$$E\{v \cdot Q\} = E[Q\{E\}^\top\{v\}]_E \quad \text{or} \quad E\{v \cdot Q\}^\top = E\{v\}^\top [E\{Q\}]_E.$$

We also have

$$v \cdot Q \cdot u = \sum_{i=1}^{3} \sum_{j=1}^{3} Q_{ij}v \cdot (\hat{e}_i\hat{e}_j) \cdot u = \sum_{i=1}^{3} \sum_{j=1}^{3} Q_{ij}(v \cdot \hat{e}_i)(\hat{e}_j \cdot u)$$

or

$$v \cdot Q \cdot u = \sum_{i=1}^{3} \sum_{j=1}^{3} Q_{ij}v_iu_j = \sum_{i=1}^{3} \sum_{j=1}^{3} v_iQ_{ij}u_j = E\{v\}^\top [E\{Q\}]_E^\top \{u\}. \quad (5.42)$$

5.12 The Inertia Dyadic

Returning now to the subject of inertia, let us consider the inertia vector

$$I_a = \sum_{i=1}^{N} m_i p_i \times (\hat{n}_a \times p_i)$$
of a system $S$ of $N$ particles for the unit vector $\hat{n}_a$, relative to O. Using the identity

$$A \times (B \times C) = (A \cdot C)B - (A \cdot B)C$$

we may write this as

$$I_a = \sum_{i=1}^{N} m_i ((p_i \cdot p_i)\hat{n}_a - (p_i \cdot \hat{n}_a)p_i) = \sum_{i=1}^{N} m_i (p_i^2\hat{n}_a - (p_i \cdot \hat{n}_a)p_i)$$

$$= \sum_{i=1}^{N} m_i (p_i^2\hat{n}_a - p_i(p_i \cdot \hat{n}_a)) = \sum_{i=1}^{N} m_i (p_i^2U \cdot \hat{n}_a - (p_i p_i) \cdot \hat{n}_a)$$

$$= \sum_{i=1}^{N} m_i (p_i^2U - p_i p_i) \cdot \hat{n}_a$$

or simply

$$I_a = I \cdot \hat{n}_a \quad (5.43a)$$

where

$$I = \sum_{i=1}^{N} m_i (p_i^2U - p_i p_i). \quad (5.43b)$$

is called the inertia dyadic of the system $S$, as measured from point O. Note that we may also write this as

$$I_a = \hat{n}_a \cdot I$$

since $U \cdot a = a \cdot U$ for all vectors $a$, and

$$(aa) \cdot b = a(a \cdot b) = (b \cdot a)a = b \cdot (aa)$$

for all vectors $a$ and $b$. Also, for a continuous system $S$, we have

$$I = \int_{\text{system}} (p^2U - pp)dm. \quad (5.43c)$$

Note that $I$ does not depend on the choice of SRT and note that

$$I_{ab} = I_a \cdot \hat{n}_b = \hat{n}_b \cdot I_a = \hat{n}_b \cdot I \cdot \hat{n}_a$$

which is the same as

$$I_{ab} = \hat{n}_a \cdot I \cdot \hat{n}_b \quad \text{since} \quad I_{ab} = I_{ba}. \quad (5.44)$$
Although the inertia dyadic $I$ is *basis independent*, it can be expressed in various basis-dependent forms. For example, if

$$A = \{\hat{n}_1, \hat{n}_2, \hat{n}_3\}$$

is a SRT, and if

$$I_j = \sum_{i=1}^{N} m_i \mathbf{p}_i \times (\hat{n}_j \times \mathbf{p}_i)$$

is the inertia vector of $S$ relative to $O$ for $\hat{n}_j$, then we have

$$I_j = \hat{n}_j \cdot I$$

so that

$$I_j \hat{n}_j = (\hat{n}_j \cdot I)\hat{n}_j$$

or

$$\sum_{j=1}^{3} I_j \hat{n}_j = \sum_{j=1}^{3} (\hat{n}_j \cdot I)\hat{n}_j = \sum_{j=1}^{3} \hat{n}_j (\hat{n}_j \cdot I).$$

(5.45)

Now we may write $I$ in terms of the basis set

$$E = \{\hat{n}_p \hat{n}_q | p = 1, 2, 3, q = 1, 2, 3\}$$

as

$$I = \sum_{p=1}^{3} \sum_{q=1}^{3} I_{pq} \hat{n}_p \hat{n}_q.$$  (5.46a)

Then

$$\sum_{j=1}^{3} I_j \hat{n}_j = \sum_{j=1}^{3} (\hat{n}_j \cdot I)\hat{n}_j = \sum_{j=1}^{3} \left(\hat{n}_j \cdot \sum_{p=1}^{3} \sum_{q=1}^{3} I_{pq} \hat{n}_p \hat{n}_q\right)\hat{n}_j$$

$$= \sum_{j=1}^{3} \sum_{p=1}^{3} \sum_{q=1}^{3} (I_{pq} \hat{n}_j \cdot (\hat{n}_p \hat{n}_q))\hat{n}_j = \sum_{j=1}^{3} \sum_{p=1}^{3} \sum_{q=1}^{3} (I_{pq} (\hat{n}_j \cdot \hat{n}_p)\hat{n}_q)\hat{n}_j$$

$$= \sum_{j=1}^{3} \sum_{p=1}^{3} \sum_{q=1}^{3} I_{pq} \delta_{jp} \hat{n}_q \hat{n}_j$$
or

\[ \sum_{j=1}^{3} I_j \hat{n}_j = \sum_{j=1}^{3} \sum_{q=1}^{3} \hat{n}_j I_{jq} \hat{n}_q = \mathcal{I}. \]

Thus we see that

\[ \mathcal{I} = \sum_{j=1}^{3} I_j \hat{n}_j. \] (5.46b)

### 5.13 Kinetic Energy And The Inertia Dyadic

Once again, consider a system of \( N \) particles having masses \( m_1, m_2, m_3, \ldots, m_N \) and located at the positions \( \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \ldots, \mathbf{p}_N \), respectively, as measured from some fixed point \( O \) in a rigid body \( A \) that contains an inertial SRT. The total kinetic energy of this system, as measured from the rigid body \( A \) is given by

\[ ^A K^S = \frac{1}{2} \sum_{i=1}^{N} m_i (\hat{\mathbf{p}}_i \cdot \hat{\mathbf{p}}_i) \quad \text{where} \quad \frac{^A d\mathbf{p}_i}{dt} = \hat{\mathbf{p}}_i = ^A \mathbf{v}_P_i \]

is the velocity of particle \( m_i \) as measured from point \( O \) in \( A \). Suppose the center-of-mass (C) of this system of \( N \) particles (as measured from point \( O \) in \( A \)) is located at the point

\[ \mathbf{r}_C = \frac{1}{m} \sum_{i=1}^{N} m_i \mathbf{p}_i \quad \text{where} \quad m = \sum_{i=1}^{N} m_i \]

is the total mass of the system of \( N \) particles. Letting \( \mathbf{r}_i \) be the position of particle \( i \) as measured from the center of mass \( C \), we have from vector addition

\[ \mathbf{p}_i = \mathbf{r}_C + \mathbf{r}_i \]

for \( i = 1, 2, 3, \ldots, N \). Then we see that

\[ \frac{^A d\mathbf{p}_i}{dt} = \frac{^A d\mathbf{r}_C}{dt} + \frac{^A d\mathbf{r}_i}{dt} \]

or

\[ \hat{\mathbf{p}}_i = \hat{\mathbf{r}}_i + \hat{\mathbf{r}}_C \]

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with \( \frac{\Lambda dr_i}{dt} = \dot{r}_i \), \( \frac{\Lambda dr_C}{dt} = \dot{r}_C \) and \( \frac{\Lambda dp_i}{dt} = \dot{p}_i \).

Then

\[
\Lambda K_S = \frac{1}{2} \sum_{i=1}^{N} m_i ((\dot{r}_C + \dot{r}_i) \cdot (\dot{r}_C + \dot{r}_i))
\]

\[
= \frac{1}{2} \sum_{i=1}^{N} m_i (\dot{r}_C \cdot \dot{r}_C + 2 (\dot{r}_C \cdot \dot{r}_i) + \dot{r}_i \cdot \dot{r}_i)
\]

\[
= \frac{1}{2} \sum_{i=1}^{N} m_i (\dot{r}_C \cdot \dot{r}_C) + \sum_{i=1}^{N} m_i (\dot{r}_C \cdot \dot{r}_i)
\]

\[
+ \frac{1}{2} \sum_{i=1}^{N} m_i (\dot{r}_i \cdot \dot{r}_i)
\]

\[
= \frac{1}{2} \left( \sum_{i=1}^{N} m_i \right) (\dot{r}_C \cdot \dot{r}_C) + \dot{r}_C \cdot \sum_{i=1}^{N} m_i \dot{r}_i
\]

\[
+ \frac{1}{2} \sum_{i=1}^{N} m_i (\dot{r}_i \cdot \dot{r}_i).
\]

But

\[
\sum_{i=1}^{N} m_i = m
\]

and we already saw that

\[
\sum_{i=1}^{N} m_i \dot{r}_i = \frac{\Lambda d}{dt} \left( \sum_{i=1}^{N} m_i r_i \right)
\]

\[
= \frac{\Lambda d}{dt} \sum_{i=1}^{N} m_i (p_i - r_C) = \frac{\Lambda d}{dt} \left( \sum_{i=1}^{N} m_i p_i - \sum_{i=1}^{N} m_i r_C \right)
\]

\[
= \frac{\Lambda d}{dt} (mr_C - mR_C) = \frac{\Lambda d0}{dt} = 0
\]

and so we have

\[
\Lambda K_S = \frac{1}{2} m (\dot{r}_C \cdot \dot{r}_C) + \frac{1}{2} \sum_{i=1}^{N} m_i (\dot{r}_i \cdot \dot{r}_i)
\]
or

\[ A^K_S = \frac{1}{2} m (\dot{r}_C \cdot \dot{r}_C) + \frac{1}{2} \sum_{i=1}^{N} m_i (\dot{p}_i - \dot{r}_C) \cdot (\dot{p}_i - \dot{r}_C) \]

Thus we are lead to Koenig’s Theorem, which states that

\[ A^K_S = A^K_1 + A^K_2 \quad (5.47a) \]

where

\[ A^K_1 = \frac{1}{2} m (\dot{r}_C \cdot \dot{r}_C) = \frac{1}{2} m |\dot{r}_C|^2 \quad (5.47b) \]

is the kinetic energy as measured from \( A \), of a single particle of mass \( m \) (equal to the system’s total mass) moving as if it were attached to the center-of-mass, and

\[ A^K_2 = \frac{1}{2} \sum_{i=1}^{N} m_i |\dot{p}_i - \dot{r}_C|^2 \quad (5.47c) \]

is the kinetic energy of the system of particles relative to the center of mass of the system, since \( |\dot{p}_i - \dot{r}_C| \) gives the speed of \( P_i \) as measured from \( C \).

Suppose now that the system \( S \) of \( N \) particles are rigidly connected so that they form a rigid body, then we know from earlier discussions that

\[ A^v_{P_i} = A^v_C + \omega^S \times r_i \quad \text{or} \quad \dot{p}_i = \dot{r}_C + \omega^S \times r_i \]

where \( \omega^S \) is the angular velocity of the system as seen by a rigid body in which the system’s center of mass \( C \) is fixed (i.e., the angular velocity of \( S \) about \( C \), as measured in \( A \)), and \( r_i \) extends from \( C \) to point \( P_i \), and putting this into the expression for \( A^K_2 \), we get

\[ A^K_2 = \frac{1}{2} \sum_{i=1}^{N} m_i |\dot{p}_i - \dot{r}_C|^2 = \frac{1}{2} \sum_{i=1}^{N} m_i (\omega^S \times r_i) \cdot (\omega^S \times r_i) \]

Using the vector identity

\[ (a \times b) \cdot (c \times d) = a \cdot (b \times (c \times d)) \]

we have

\[ (\omega^S \times r_i) \cdot (\omega^S \times r_i) = \omega^S \cdot (r_i \times (\omega^S \times r_i)) \]
and so
\[ \mathbf{A} \mathbf{K}^S_2 = \frac{1}{2} \sum_{i=1}^{N} m_i \mathbf{C}_i \mathbf{w}^S \cdot (\mathbf{r}_i \times (\mathbf{C}_i \omega^S \times \mathbf{r}_i)) \]
or
\[ \mathbf{A} \mathbf{K}^S_2 = \frac{1}{2} \mathbf{C}_i \mathbf{w}^S \cdot \sum_{i=1}^{N} m_i (\mathbf{r}_i \times (\mathbf{C}_i \omega^S \times \mathbf{r}_i)). \] (5.48)

From the vector identity
\[ \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \]
and the above discussions on dyads, we may write
\[ \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{ca} \cdot \mathbf{b}) \]
or
\[ \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b})\mathbf{U} \cdot \mathbf{b} - (\mathbf{ca} \cdot \mathbf{b}) \]
where \( \mathbf{U} \) is the unit dyadic. Thus we have
\[ \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = ((\mathbf{a} \cdot \mathbf{b})\mathbf{U} - \mathbf{ca}) \cdot \mathbf{b}. \]

Using this we now write
\[ \mathbf{r}_i \times (\mathbf{C}_i \mathbf{w}^S \times \mathbf{r}_i) = ((\mathbf{r}_i \cdot \mathbf{r}_i)\mathbf{U} - \mathbf{r}_i \mathbf{r}_i) \cdot (\mathbf{C}_i \mathbf{w}^S) \]
and so
\[ \sum_{i=1}^{N} m_i (\mathbf{r}_i \times (\mathbf{C}_i \mathbf{w}^S \times \mathbf{r}_i)) = \sum_{i=1}^{N} m_i ((\mathbf{r}_i \cdot \mathbf{r}_i)\mathbf{U} - \mathbf{r}_i \mathbf{r}_i) \cdot (\mathbf{C}_i \mathbf{w}^S) \]
or
\[ \sum_{i=1}^{N} m_i (\mathbf{p}_i \times (\mathbf{C}_i \mathbf{w}^S \times \mathbf{r}_i)) = \mathbf{C}_i \mathbf{I}_S^S \cdot \mathbf{C}_i \mathbf{w}^S \]
where
\[ \mathbf{C}_i \mathbf{I}_S^S = \sum_{i=1}^{N} m_i ((\mathbf{r}_i \cdot \mathbf{r}_i)\mathbf{U} - \mathbf{r}_i \mathbf{r}_i) \] (5.49a)
is the moment of inertia dyadic of the rigid system \( S \) of \( N \) particles about the system’s center of mass \( \mathbf{C} \). If the system is a continuous rigid body with mass element \( dm \), then we write
\[ \mathbf{C}_i \mathbf{I}_S^S = \int_{\text{system}} ((\mathbf{r} \cdot \mathbf{r})\mathbf{U} - \mathbf{rr}) dm \] (5.49b)

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where the integration is taken over the entire system $S$. Using $^C\mathcal{I}^S$ we now may write Equation (5.48) as

$$^A K_2^S = \frac{1}{2} C\omega^S \cdot ^C\mathcal{I}^S \cdot C\omega^S.$$  \hfill (5.50)

The total kinetic energy of a rigid system $S$ (with center of mass $C$) as measured by an inertial reference frame $A$ can now be written as

$$^A K^S = \frac{1}{2} m|\dot{r}_C|^2 + \frac{1}{2} C\omega^S \cdot ^C\mathcal{I}^S \cdot C\omega^S$$ \hfill (5.51)

where:

- $|\dot{r}_C|$ is the speed of the system’s center of mass as measured by $A$,
- $m$ is the total mass of the system,
- $C\omega^S$ is the angular velocity of the system as seen by a rigid body in which the system’s center of mass is fixed (i.e., the angular velocity of $S$ about point $C$, as measured in $A$), and
- $^C\mathcal{I}^S$ is the moment of inertia dyadic of the system $S$ as measured from its center of mass point $C$.

### 5.14 Angular Momentum And The Inertia Dyadic

Once again, consider a system $S$ of $N$ particles having masses $m_1, m_2, m_3, \ldots, m_N$ and located at the positions $p_1, p_2, p_3, \ldots, p_N$, respectively, as measured from some fixed point $O$ in a rigid body $A$. The total angular momentum of the system about point $O$, as measured by $A$ is given by

$$^A \mathbf{H}_O = \sum_{i=1}^N p_i \times m_i \dot{p}_i.$$ \hfill (5.52)

Suppose the center-of-mass of this system of $N$ particles is located at the point

$$\mathbf{r}_C = \frac{1}{m} \sum_{i=1}^N m_i \mathbf{p}_i \quad \text{where} \quad m = \sum_{i=1}^N m_i$$
is the total mass of the system of $N$ particles. Letting $r_i$ be the position of particle $P_i$ as measured from the center of mass $C$, we have from vector addition

$$p_i = r_C + r_i$$

for $i = 1, 2, 3, \ldots, N$. Then we see that

$$\frac{A}{dt} dp_i = \frac{A}{dt} dr_C + \frac{A}{dt} dr_i$$

or

$$\dot{p}_i = \dot{r}_i + \dot{r}_C$$

with

$$\frac{A}{dt} dr_i = \dot{r}_i, \quad \frac{A}{dt} dr_C = \dot{r}_C$$

and

$$\frac{A}{dt} dp_i = \dot{p}_i,$$

which then leads to

$$^A H_0 = \sum_{i=1}^N m_i (r_i + r_C) \times (\dot{r}_i + \dot{r}_C)$$

$$= \sum_{i=1}^N m_i (r_i \times \dot{r}_i + r_i \times \dot{r}_C + r_C \times \dot{r}_i + r_C \times \dot{r}_C)$$

$$= \sum_{i=1}^N m_i (r_i \times \dot{r}_i) + \left( \sum_{i=1}^N m_i r_i \right) \times \dot{r}_C + r_C \times \left( \sum_{i=1}^N m_i \dot{r}_i \right) + \left( \sum_{i=1}^N m_i \right) r_C \times \dot{r}_C$$

But recall that

$$\sum_{i=1}^N m_i r_i = 0 \quad \text{and} \quad \sum_{i=1}^N m_i \dot{r}_i = 0 \quad \text{and} \quad \sum_{i=1}^N m_i = m$$

and so we have

$$^A H_0 = \sum_{i=1}^N m_i (r_i \times \dot{r}_i) + m (r_C \times \dot{r}_C)$$

or, since

$$\frac{A}{dt} dp_i = ^A v^p_i - ^A v^C$$

we have

$$^A H_0 = \bar{r} \times (m^A v^C) + \sum_{i=1}^N r_i \times m_i \dot{r}_i$$

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and so we have the result

\[ A^H_O = A^H_{O1} + A^H_{O2} \]  \hspace{1cm} (5.53a)

where

\[ A^H_{O1} = r_C \times m \dot{r}_C \]  \hspace{1cm} (5.53b)

is the angular momentum (as measured in \( A \)) about point O of a single particle of mass \( m \) (equal to the system’s total mass) moving as if it were attached to the center-of-mass, and

\[ A^H_{O2} = \sum_{i=1}^{N} r_i \times m_i \dot{r}_i \]  \hspace{1cm} (5.53c)

is the angular momentum of the system of particles relative to the center-of-mass of the system. Note that since \( \dot{r}_i = \dot{\mathbf{p}}_i - \dot{r}_C \), we may write Equation (5.53c) also as

\[ A^H_{O2} = \sum_{i=1}^{N} r_i \times m_i \dot{\mathbf{p}}_i \]  \hspace{1cm} (5.53d)

or just

\[ A^H_{O2} = \sum_{i=1}^{N} r_i \times m_i \dot{\mathbf{p}}_i. \]

Suppose now that the system \( S \) of \( N \) particles are rigidly connected and suppose that \( S \) is rotating with angular velocity \( \omega^S \), as measured by a rigid body in which \( C \) is fixed. Then we know from earlier discussions that

\[ A^v_{P_i} = A^v_C + C^\omega^S \times r_i \] \hspace{1cm} or \hspace{1cm} \[ \dot{\mathbf{p}}_i = \dot{\mathbf{r}}_C + C^\omega^S \times \mathbf{r}_i \]
where $C_\omega S$ is the angular velocity of the system as seen by a rigid body in which the system's center of mass $C$ is fixed, and $r_i$ extends from $C$ to point $P_i$, and putting this into the expression for $^A\mathbf{H}_{O2}$, we get

$$^A\mathbf{H}_{O2} = \sum_{i=1}^{N} (r_i \times m_i \dot{p}_i) = \sum_{i=1}^{N} r_i \times m_i (\dot{r}_C + \,^C_\omega S \times r_i)$$

$$= \left( \sum_{i=1}^{N} m_i r_i \right) \times \dot{r}_C + \sum_{i=1}^{N} r_i \times m_i (\,^C_\omega S \times r_i)$$

or

$$^A\mathbf{H}_{O2} = \sum_{i=1}^{N} m_i (r_i \times (\,^C_\omega S \times r_i))$$

Using the vector identity

$$a \times (b \times c) = ((c \cdot a)\mathbf{U} - c a) \cdot b$$

as derived in the previous section, we write

$$r_i \times (\,^C_\omega S \times r_i) = ((r_i \cdot r_i)\mathbf{U} - r_i r_i) \cdot \,^C_\omega S$$

and so

$$^A\mathbf{H}_{O2} = \sum_{i=1}^{N} r_i \times m_i (\,^C_\omega S \times r_i) = \sum_{i=1}^{N} m_i ((r_i \cdot r_i)\mathbf{U} - r_i r_i) \cdot \,^C_\omega S$$

or

$$^A\mathbf{H}_{O2} = \,^C\mathbf{I}_S \cdot \,^C_\omega S$$

where

$$\,^C\mathbf{I}_S = \sum_{i=1}^{N} m_i ((r_i \cdot r_i)\mathbf{U} - r_i r_i)$$

is what we earlier called the moment of inertia dyadic of the rigid system $S$ of $N$ particles about the system’s center of mass. If the system is a continuous rigid body with mass element $dm$, then we write

$$\,^C\mathbf{I}_S = \int_{\text{system}} ((r \cdot r)\mathbf{U} - \mathbf{r} \mathbf{r}) dm$$
where the integration is over the entire rigid body.

The total angular momentum of a rigid system $S$ (with center of mass $C$) as measured by an inertial reference frame $A$ can now be written as

$$^A\mathbf{H}_O = \mathbf{r}_C \times m \dot{\mathbf{r}}_C + ^C\mathbf{T}^S \cdot ^C\omega^S$$  \hspace{1cm} (5.54)

where:

- $\mathbf{r}_C$ is the position of the center of mass $C$ relative to point $O$ as measured by $A$,
- $\dot{\mathbf{r}}_C$ is the velocity of the system’s center of mass as measured by $A$,
- $m$ is the total mass of the system,
- $^C\omega^S$ is the angular velocity of the system as seen by a rigid body in which the system’s center of mass is fixed, and
- $^C\mathbf{T}^S$ is the moment of inertia dyadic of the system as measured from its center of mass point $C$.

**Simplification for the Center-Of-Mass Point**

Suppose next that we consider a system $S$ of $N$ particles having masses $m_1$, $m_2$, $m_3$, ..., $m_N$ and located at the positions $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, ..., $\mathbf{r}_N$, respectively, as measured from the center of mass point $C$, then Equation (5.54) with reduces to having $\mathbf{r}_C = \mathbf{0}$, and

$$^A\mathbf{H}_C = ^C\mathbf{T}^S \cdot ^C\omega^S$$  \hspace{1cm} (5.55a)

**Simplification for a Fixed Point**

Suppose next that we consider a system $S$ of $N$ particles having masses $m_1$, $m_2$, $m_3$, ..., $m_N$ and located at the positions $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, ..., $\mathbf{p}_N$, respectively, as measured from some fixed point $O$ in a rigid body $A$. Once again, the total angular momentum of the system about point $O$, as measured by $A$ is given by

$$^A\mathbf{H}_O = \sum_{i=1}^{N} \mathbf{p}_i \times m_i \dot{\mathbf{p}}_i.$$
Suppose now that the entire system \( S \) lies on a rigid body \( B \) that rotates with angular velocity \( A^B \) about the fixed point \( O \) in \( A \). Then we know from earlier discussions that
\[
\dot{p}_i = A^B \times p_i
\]
where \( p_i \) extends from \( O \) to point \( P_i \), and putting this into the expression for \( A^H O \), we get
\[
A^H O = \sum_{i=1}^{N} p_i \times m_i \dot{p}_i = \sum_{i=1}^{N} p_i \times m_i (A^B \times p_i)
\]
or
\[
A^H O = \sum_{i=1}^{N} m_i (p_i \times (A^B \times p_i))
\]
Using the vector identity
\[
a \times (b \times c) = ((c \cdot a)U - ca) \cdot b
\]
as derived in the previous section, we write
\[
p_i \times (A^B \times p_i) = ((p_i \cdot p_i)U - p_i p_i) \cdot A^B
\]
and so
\[
A^H O = \sum_{i=1}^{N} m_i ((p_i \cdot p_i)U - p_i p_i) \cdot A^B
\]
or
\[
A^H O = O^T S \cdot A^B
\]
where
\[
O^T S = \sum_{i=1}^{N} m_i ((p_i \cdot p_i)U - p_i p_i)
\]
is what we earlier called the moment of inertia dyadic of the system \( S \) of \( N \) particles that is fixed to the rigid body \( B \). If the system is a continuous rigid body with mass element \( dm \), then we write
\[
O^T S = \int_{\text{system}} ((p \cdot p)U - pp)dm
\]
where the integration is over the entire rigid body.

The total angular momentum of a rigid system $S$ that is fixed to a rigid body $B$ that rotates with angular velocity $\mathbf{A} \omega^B$ about a fixed point $O$ that is fixed in an inertial reference frame $A$ can now be written as

$$
\mathbf{A} \mathbf{H}_O = \mathbf{O}^S \cdot \mathbf{A} \omega^B
$$

(5.55b)

5.15 The Shifting Theorem (Parallel Axis Theorem) For $\mathbf{O}^S$

Starting with

$$
\mathbf{O}^S = \sum_{i=1}^{N} m_i ((\mathbf{p}_i \cdot \mathbf{p}_i) \mathbf{U} - \mathbf{p}_i \mathbf{p}_i)
$$

where $\mathbf{p}_i$ are measured from point $O$, and suppose that the system’s center-of-mass $C$, is

$$
\mathbf{r}_C = \frac{1}{m} \sum_{i=1}^{N} m_i \mathbf{p}_i,
$$

are measured from some point $O$ fixed in $A$. Then writing

$$
\mathbf{p}_i = \mathbf{r}_i + \mathbf{r}_C
$$

where $\mathbf{r}_i$ is the position of $P_i$ from the mass center $C$, we have

$$
\mathbf{O}^S = \sum_{i=1}^{N} m_i ((\mathbf{r}_i + \mathbf{r}_C) \cdot (\mathbf{r}_i + \mathbf{r}_C)) \mathbf{U} - (\mathbf{r}_i + \mathbf{r}_C)(\mathbf{r}_i + \mathbf{r}_C))
$$

$$
= \sum_{i=1}^{N} m_i (\mathbf{r}_i \cdot \mathbf{r}_i + 2 \mathbf{r}_C \cdot \mathbf{r}_i + \mathbf{r}_C \cdot \mathbf{r}_C) \mathbf{U}
$$

$$
- \sum_{i=1}^{N} m_i (\mathbf{r}_i \mathbf{r}_i + \mathbf{r}_C \mathbf{r}_i + \mathbf{r}_i \mathbf{r}_C + \mathbf{r}_C \mathbf{r}_C)
$$

$$
= \sum_{i=1}^{N} m_i ((\mathbf{r}_i \cdot \mathbf{r}_i) \mathbf{U} - \mathbf{r}_i \mathbf{r}_i) + 2 \left( \mathbf{r}_C \cdot \sum_{i=1}^{N} m_i \mathbf{r}_i \right) \mathbf{U}
$$

$$
+ \left( \sum_{i=1}^{N} m_i \right) (\mathbf{r}_C \cdot \mathbf{r}_C) \mathbf{U} - \mathbf{r}_C \left( \sum_{i=1}^{N} m_i \mathbf{r}_i \right) - \left( \sum_{i=1}^{N} m_i \mathbf{r}_i \right) \mathbf{r}_C
$$
\[- \left( \sum_{i=1}^{N} m_i \right) (\mathbf{r}_C \mathbf{r}_C) \cdot \]

But, as demonstrated previously in these notes,

\[ \sum_{i=1}^{N} m_i = m \quad \text{and} \quad \sum_{i=1}^{N} m_i \mathbf{r}_i = \mathbf{0} \]

and so we get

\[ O^S = \sum_{i=1}^{N} m_i (\mathbf{r}_i \cdot \mathbf{r}_i) \mathbf{U} - \mathbf{r}_i \mathbf{r}_i) + 2(\mathbf{r}_C \cdot \mathbf{0}) \mathbf{U} + m(\mathbf{r}_C \cdot \mathbf{r}_C) \mathbf{U} - \mathbf{r}_C \mathbf{0} - \mathbf{0r}_C - m(\mathbf{r}_C \mathbf{r}_C) \]

or simply

\[ O^S = O^C + \mathcal{I}^S \]

(5.56a)

where

\[ \mathcal{I}^S = \sum_{i=1}^{N} m_i ((\mathbf{r}_i \cdot \mathbf{r}_i) \mathbf{U} - \mathbf{r}_i \mathbf{r}_i) \]

(5.56b)

is the inertia dyadic of the system \( S \) relative to the system’s center-of-mass \( C \) and

\[ O^C = m((\mathbf{r}_C \cdot \mathbf{r}_C) \mathbf{U} - \mathbf{r}_C \mathbf{r}_C) \]

(5.56c)

is the inertia dyadic about point \( O \) of a single (fictitious) particle of mass \( m \) located at the system’s center of mass \( C \). This expression is known as the shifting theorem for \( O^S \) and is a generalization of the parallel-axis theorem you may have seen in elementary physics.

In matrix form using \( E \) as a SRT along the \( xyz \) axes, the shifting theorem reads

\[ [E(O^S)]_E = [E(O^C)]_E + [E(\mathcal{I}^S)]_E \]

(5.57a)

with

\[ [E(O^C)]_E = (x^2 + \bar{y}^2 + \bar{z}^2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} x_C \\ y_C \\ z_C \end{bmatrix} \begin{bmatrix} x_C & y_C & z_C \end{bmatrix} \]
or

\[ [E(OI)]= \begin{bmatrix} y^2 + z^2 & -xyz & -x^2z \\ -x^2y & x^2 + z^2 & -yz \\ -x^2z & -y^2z & x^2 + y^2 \end{bmatrix} \]

and so

\[ [E(OI)] = [E(CO)] + m \begin{bmatrix} y^2 + z^2 & -x^2y & -xz \\ -x^2y & x^2 + z^2 & -yz \\ -x^2z & -y^2z & x^2 + y^2 \end{bmatrix} \]

Of course, once we have Equation (5.56a), then we also have

\[ \hat{n}_a \cdot OI = \hat{n}_a \cdot OC + \hat{n}_a \cdot CS \]

or

\[ OI_a = OC + CS \]

and

\[ OI_a \cdot \hat{n}_b = OC \cdot \hat{n}_b + CS \cdot \hat{n}_b \]

or

\[ OI_{ab} = OC_{ab} + CS_{ab} \]  \hspace{1cm} (5.56e)

Example #16: A Uniform Box of Lengths, a, b and c About C

Compute the inertial matrix \( I_C \) (relative to its center of mass \( C \)) for a uniform box in the region \( 0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c \). To solve this we use the result in Example #8 along with the fact that the center of mass for the box, as measured by point \( O \), is located at the point

\[ r_C = \frac{1}{2}(a\hat{e}_x + b\hat{e}_y + c\hat{e}_z). \]

Then Equation (5.57a) gives

\[ [E(OI)] = [E(CI)] + m \begin{bmatrix} (b/2)^2 + (c/2)^2 & -(a/2)(b/2) & -(a/2)(c/2) \\ -(a/2)(b/2) & (a/2)^2 + (c/2)^2 & -(b/2)(c/2) \\ -(a/2)(c/2) & -(b/2)(c/2) & (a/2)^2 + (b/2)^2 \end{bmatrix} \]

or

\[ [E(OI)] = [E(CI)] + \frac{1}{4}m \begin{bmatrix} b^2 + c^2 & -ab & -ac \\ -ab & a^2 + c^2 & -bc \\ -ac & -bc & a^2 + b^2 \end{bmatrix} \]
which says that

\[
[E(CI^S)_E] = [E(0I^S)_E] - \frac{1}{4}m \begin{bmatrix}
  b^2 + c^2 & -ab & -ac \\
  -ab & a^2 + c^2 & -bc \\
  -ac & -bc & a^2 + b^2
\end{bmatrix}
\]

We computed in Example #8 the result

\[
[E(0I^S)_E] = \frac{1}{12}m \begin{bmatrix}
  4(b^2 + c^2) & -3ab & -3ac \\
  -3ab & 4(a^2 + c^2) & -3bc \\
  -3ac & -3bc & 4(a^2 + b^2)
\end{bmatrix}
\]

and so

\[
[E(CI^S)_E] = \frac{1}{12}m \begin{bmatrix}
  4(b^2 + c^2) & -3ab & -3ac \\
  -3ab & 4(a^2 + c^2) & -3bc \\
  -3ac & -3bc & 4(a^2 + b^2)
\end{bmatrix} - \frac{1}{4}m \begin{bmatrix}
  b^2 + c^2 & -ab & -ac \\
  -ab & a^2 + c^2 & -bc \\
  -ac & -bc & a^2 + b^2
\end{bmatrix}
\]

which reduces to

\[
[E(CI^S)_E] = \frac{1}{12}m \begin{bmatrix}
  b^2 + c^2 & 0 & 0 \\
  0 & a^2 + c^2 & 0 \\
  0 & 0 & a^2 + b^2
\end{bmatrix}.
\] (5.58)

Example #17: Kinetic Energy

Compute the kinetic energy of rotation for a uniform box in the region \(0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c\), if it rotates about its center of mass with angular velocity

\[
\omega = \omega_x \hat{e}_x + \omega_y \hat{e}_y + \omega_z \hat{e}_z
\]

with SRT \(E = \{\hat{e}_x, \hat{e}_y, \hat{e}_z\}\).

To solve this we use Equation (5.58),

\[
[E(CI^S)_E] = \frac{1}{12}m \begin{bmatrix}
  b^2 + c^2 & 0 & 0 \\
  0 & a^2 + c^2 & 0 \\
  0 & 0 & a^2 + b^2
\end{bmatrix}
\]
from Example #16, and the use Equation (5.50),

\[ A^SK_2 = \frac{1}{2} C^S \cdot C^T \cdot C^S. \]

in matrix form,

\[ A^SK_2 = \frac{1}{2} E \{ C^S \}^T \left[ E \{ C^T \} \right] E \{ C^S \}, \]

to get

\[ A^SK_2 = \frac{1}{2} \left\{ \omega_x \ \omega_y \ \omega_z \right\} \frac{1}{12} m \begin{bmatrix} b^2 + c^2 & 0 & 0 \\ 0 & a^2 + c^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{bmatrix} \left\{ \omega_x \omega_y \omega_z \right\} \]

or

\[ A^SK_2 = \frac{1}{24} m \left\{ (b^2 + c^2)\omega_x^2 + (a^2 + c^2)\omega_y^2 + (a^2 + b^2)\omega_z^2 \right\}. \]

5.16 The Transformation Of Dyadics Under Rotations

Suppose that a rigid body \( B \) with SRT

\[ B = \{ b_1, b_2, b_3 \} \]

rotates as measured from some inertial rigid body \( A \) with SRT, \( A = \{ a_1, a_2, a_3 \}. \)

We know that if \( u \) is any vector fixed in \( B \), then

\[ A\{u\} = [A^RB]^B\{u\} \]

where \( A\{u\} \) and \( B\{u\} \) are the coordinates of \( u \) with respect to the standard reference triads

\[ A = \{ a_1, a_2, a_3 \} \quad \text{and} \quad B = \{ b_1, b_2, b_3 \}, \]

is \( A \) and \( B \), respectively, and \([A^RB]\) is the rotation matrix with elements

\[ [A^RB]_{ij} = a_i \cdot b_j. \]

We also know that if \( Q \) is any dyadic and if \( u \) is any vector, then

\[ A\{Q \cdot u\} = [A^QA]^A\{u\} \quad \text{and} \quad B\{Q \cdot u\} = [B^QB]^B\{u\}. \]
Putting the expression $^A\{u\} = [^A R_B]^B\{u\}$ in the first of these equations leads to

$^A\{Q \cdot u\} = [^A Q_A]^A\{u\} = [^A Q_A][^A R_B]^B\{u\}$

or, since $^A\{Q \cdot u\} = [^A R_B]^B\{Q \cdot u\}$, we have

$[^A R_B]^B\{Q \cdot u\} = [^A Q_A][^A R_B]^B\{u\}$

so that

$^B\{Q \cdot u\} = [^A R_B]^{-1}[^A Q_A][^A R_B]^B\{u\} = [^A R_B]^T[^A Q_A][^A R_B]^B\{u\}$

Thus we see that

$[^B Q_B]^B\{u\} = [^A R_B]^{-1}[^A Q_A][^A R_B]^B\{u\}$

and (since $u$ was arbitrary), we have

$[^B Q_B] = [^A R_B]^{-1}[^A Q_A][^A R_B]$ 

or

$[^B Q_B] = [^A R_B]^T[^A Q_A][^A R_B] = [^B R_A][^A Q_A][^A R_B]$  \hspace{1cm} (5.59)


### 5.17 Principal Axes And Principal Moments Of Inertia

Suppose that a system $S$ is fixed to rigid body $B$ that rotates about some fixed point $O$ as measured from some inertial rigid body $A$. We had seen that the angular momentum of $S$ about point $O$ as computed in $A$ is given by the expression

$^A H_O = ^0 T^S \cdot ^A \omega^B$

where $^A \omega^B$ is the angular velocity of $B$ as measured by $A$ and $^0 T^S$ is the moment of inertia dyadic of $S$ about some point $O$ in $A$. In general, we see that $H_O$ and $^A \omega^B$ are not parallel.

If $^0 T^S$ (or simply $T$) is the moment of inertia dyadic of a system $S$ about some point $O$, a unit vector $u$ is called a principal axis of inertia if $T \cdot u$ is parallel to $u$. The moment of inertia with respect to a principle axis, $u \cdot T \cdot u$, is called a
**principal moment of inertia** about \( \mathbf{u} \). To determine principal axes and principal moments of inertia, we start with \( \mathcal{I} \cdot \mathbf{u} \) being parallel to \( \mathbf{u} \), which says that

\[
\mathcal{I} \cdot \mathbf{u} = I \mathbf{u}
\]

for some scalar \( I \). If \( E = \{ \hat{e}_1, \hat{e}_2, \hat{e}_3 \} \) is a SRT in \( A \), then

\[
E \{ \mathcal{I} \cdot \mathbf{u} \} = E \{ I \mathbf{u} \} \quad \text{or} \quad [E \mathcal{I}_E] E \{ \mathbf{u} \} = I E \{ \mathbf{u} \}
\]

which leads to

\[
([E \mathcal{I}_E] - I [E \mathcal{U}_E]) E \{ \mathbf{u} \} = \{ 0 \}
\]

resulting in

\[
\det([E \mathcal{I}_E] - I [E \mathcal{U}_E]) = \det \begin{bmatrix} I_{11} - I & I_{12} & I_{13} \\ I_{21} & I_{22} - I & I_{23} \\ I_{31} & I_{32} & I_{33} - I \end{bmatrix} = 0. \tag{5.61a}
\]

Since \([E \mathcal{I}_E]^T = [E \mathcal{I}_E]\), it can easily be shown that this equation will lead to three \textit{positive} real solutions \( I_1, I_2, I_3 \) (which may not all be distinct). For each \( I_k \), the non-zero solutions to

\[
([E \mathcal{I}_E] - I_k [E \mathcal{U}_E]) E \{ \mathbf{u}_k \} = \{ 0 \} \tag{5.61b}
\]

give the principal axis (or axes) corresponding to the principal moment of inertia \( I_i \). Since \([E \mathcal{I}_E]^T = [E \mathcal{I}_E]\), it can also be easily shown that for \( I_i \neq I_j \), we have

\[
\mathbf{u}_i \cdot \mathbf{u}_j = 0.
\]

In general, the following properties are always true regarding the inertia matrix, and this is for a system \( S \) about any point \( O \) (fixed or not fixed) in \( A \).

(a) The inertia matrix is \textit{symmetric} so that \([E \mathcal{I}_E]^T = [E \mathcal{I}_E]\).

(b) The inertia matrix is \textit{positive definite} so that all principal moments of inertia are \textit{positive}.

(c) Principal Moments of Inertia are the least when the point \( O \) is the center-of-mass point.

(d) It is always possible to find three distinct mutually orthogonal principal axes of inertia.
(e) If the reference triad in $A$ is aligned with the principal axes, the products of inertia are zero and the moments of inertia are the principal moments of inertia.

(f) The Principal Moments of Inertia do not depend on the choice of the SRT $E$ in $A$.

5.18 The Ellipsoid Of Inertia

A convenient method of representing the rotational inertia characteristics of a rigid body about some point $O$ is by means of its ellipsoid of inertia, which is essentially a plot of the moment of inertia of a body for all possible axis orientations through the reference point $O$. In order to see how the inertia characteristics are represented by an ellipsoid, consider a system $S$ fixed in a rigid body $B$ which is rotating about a fixed point $O$ as measured from some reference triad $E$ in a rigid body $A$. Note that the system $S$ could simply be the rigid body $B$, but it need not be. We have seen that the kinetic energy of this body is given by

$$A K_O = \frac{1}{2} A \omega_B \cdot O I_S A \omega_B .$$

or in matrix form

$$A K_O = \frac{1}{2} E \{ A \omega_B \}^T E (O I_S) E \{ A \omega_B \} E .$$

Since $A K_O$ is a constant for many problems, we may write this as

$$\left\{ \frac{A \omega_B}{\sqrt{2(A K_O)}} \right\}^T E (O I_S) E \left\{ \frac{A \omega_B}{\sqrt{2(A K_O)}} \right\} = 1$$

Setting

$$\left\{ \frac{A \omega_B}{\sqrt{2(A K_O)}} \right\} E = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

we get

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}^T \begin{bmatrix} I_{\alpha \alpha} & I_{\alpha \beta} & I_{\alpha \gamma} \\ I_{\beta \alpha} & I_{\beta \beta} & I_{\beta \gamma} \\ I_{\gamma \alpha} & I_{\gamma \beta} & I_{\gamma \gamma} \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 1$$
or

\[ I_{\alpha\alpha}\alpha^2 + I_{\beta\beta}\beta^2 + I_{\gamma\gamma}\gamma^2 + 2I_{\alpha\beta}\alpha\beta + 2I_{\alpha\gamma}\alpha\gamma + 2I_{\beta\gamma}\beta\gamma = 1 \]

which is an equation of an ellipsoidal surface centered at point O and this is known as the *ellipsoid of inertia* for the system \( S \) fixed in the rigid body \( B \). If the reference triad

\[ E = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\} \]

are the directions of the principal axes and if \( ^A\omega^B \) is in one of these directions, then the above equation becomes

\[
\begin{bmatrix}
\alpha' \\
\beta' \\
\gamma'
\end{bmatrix}^T
\begin{bmatrix}
I_\alpha & 0 & 0 \\
0 & I_\beta & 0 \\
0 & 0 & I_\gamma
\end{bmatrix}
\begin{bmatrix}
\alpha' \\
\beta' \\
\gamma'
\end{bmatrix} = 1
\]

or simply

\[ I_\alpha(\alpha')^2 + I_\beta(\beta')^2 + I_\gamma(\gamma')^2 = 1 \]

and then we see that the principal axes are simply perpendicular to this ellipsoid of inertia at their points of intersection.