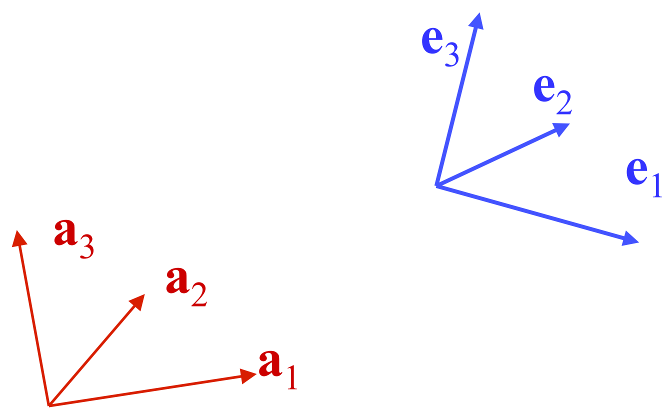
Vectors in Three Dimensions and Transformations

Scalar and Vector Functions

$\phi(q_1, q_2,\ldots,q_n)$ is a scalar function of n variables

$\phi(q_1, q_2,\ldots,q_n)$ is independent of reference frames – scalar invariant



NB: No assumptions of orthogonality!

$\mathbf{v}(q_1, q_2,\ldots,q_n)$ is a vector function $\mathbf{v}$ of n variables.

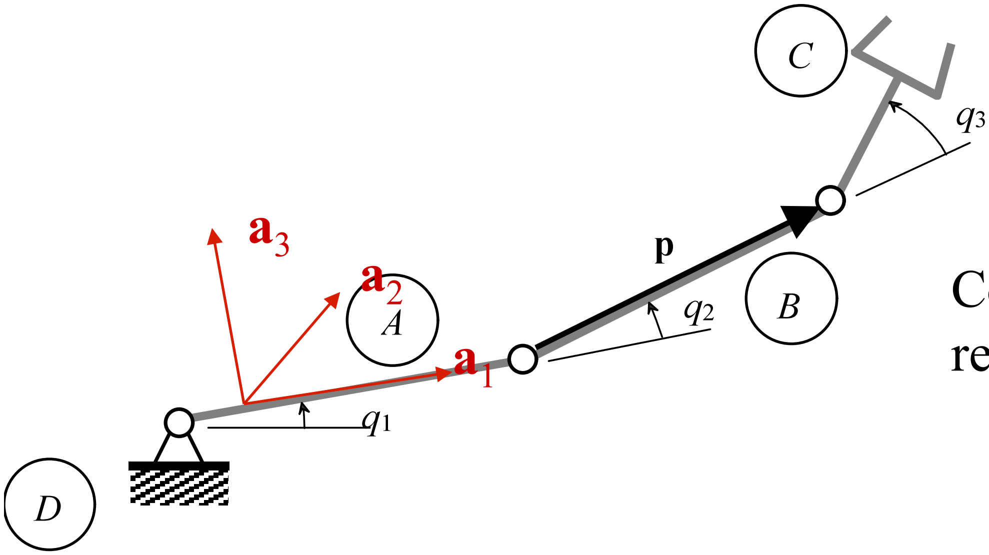
In any reference frame $\{A\}$, we can find three linearly independent (LI) vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ that are basis vectors.

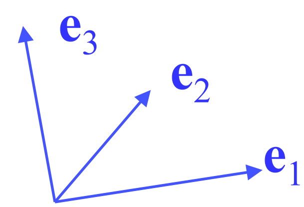
The vector function $\mathbf{v}(q_1, q_2,\ldots,q_n)$ can be expressed as a linear combination of the three vectors:

$$\mathbf{v}(q_1, q_2,\ldots,q_n) = v_1(q_1, q_2,\ldots,q_n)\,\mathbf{a}_1 + v_2(q_1, q_2,\ldots,q_n)\,\mathbf{a}_2 + v_3(q_1, q_2,\ldots,q_n)\,\mathbf{a}_3$$

The three coefficients are the three scalar functions v_1, v_2, and v_3. They are called *components* and these three functions are unique once the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are specified.

Reference Frames





Components of vectors depend on the reference triad

$$\mathbf{p} = p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + p_3\mathbf{e}_3,$$

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$$

$${}^E[\mathbf{p}] = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}, {}^E[\mathbf{v}] = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

A robotic arm is a system of rigid bodies (reference frames) *A*, *B*, and *C*. *D* is the inertial or the laboratory reference frame that is considered fixed.

$${}^E[\mathbf{p}] \neq {}^A[\mathbf{p}]$$

Position, Velocity and Acceleration Vectors

$\mathbf{p}$ is a position vector of P in A

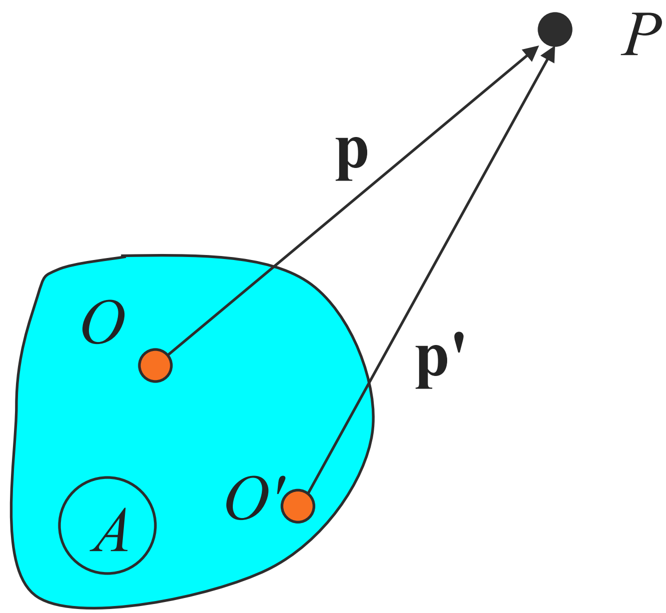
- Emanates from a point fixed to A
- Ends up at P

${}^{A}\mathbf{v}^{P}$ is the velocity of P in A

$$ {}^{A}\mathbf{v}^{P} = \frac{{}^{A}d\mathbf{p}}{dt} $$

${}^{A}\mathbf{a}^{P}$ is the acceleration of P in A

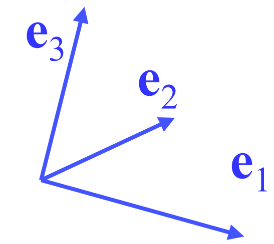
$$ {}^{A}\mathbf{a}^{P} = \frac{{}^{A}d\left({}^{A}\mathbf{v}^{P}\right)}{dt} $$

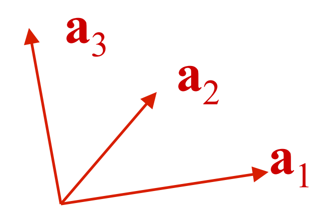


What if a different position vector were chosen?

Standard Reference Triad

Three LI vectors rigidly attached to a reference frame satisfying the equations below constitute a standard reference triad or simply a reference triad.





$$\mathbf{a}_i \cdot \mathbf{a}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

$$\mathbf{a}_1 \times \mathbf{a}_2 = \mathbf{a}_3,$$

$$\mathbf{a}_2 \times \mathbf{a}_3 = \mathbf{a}_1,$$

$$\mathbf{a}_3 \times \mathbf{a}_1 = \mathbf{a}_2$$

- Projection rule

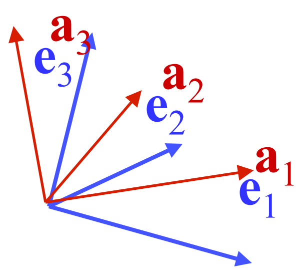
$$\mathbf{u} \cdot \mathbf{e}_i = u_i$$

- Composition rule

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 = \sum_{i=1}^{3} u_i \mathbf{e}_i$$

$$\mathbf{u} = \sum_{i=1}^{3} (\mathbf{u} \cdot \mathbf{e}_i)\mathbf{e}_i$$

Transformation of Vectors



Components in $\{A\}$ and in $\{E\}$ are related by:

$$^{E}[\mathbf{p}]_i = \sum_j (\mathbf{e}_i \cdot \mathbf{a}_j)\, ^{A}[\mathbf{p}]_j$$

$$\sum_i {}^{E}[\mathbf{p}]_i \mathbf{e}_i = \sum_j {}^{A}[\mathbf{p}]_j \mathbf{a}_j$$

$$^{E}[\mathbf{p}] = \left[{}^{E}\mathbf{R}_A \right] {}^{A}[\mathbf{p}]$$

$$= \begin{bmatrix} \mathbf{e}_1 \cdot \mathbf{a}_1 & \mathbf{e}_1 \cdot \mathbf{a}_2 & \mathbf{e}_1 \cdot \mathbf{a}_3 \\ \mathbf{e}_2 \cdot \mathbf{a}_1 & \mathbf{e}_2 \cdot \mathbf{a}_2 & \mathbf{e}_2 \cdot \mathbf{a}_3 \\ \mathbf{e}_3 \cdot \mathbf{a}_1 & \mathbf{e}_3 \cdot \mathbf{a}_2 & \mathbf{e}_3 \cdot \mathbf{a}_3 \end{bmatrix} {}^{A}[\mathbf{p}]$$

Rotation matrix that transforms components in $\{A\}$ into components in $\{E\}$

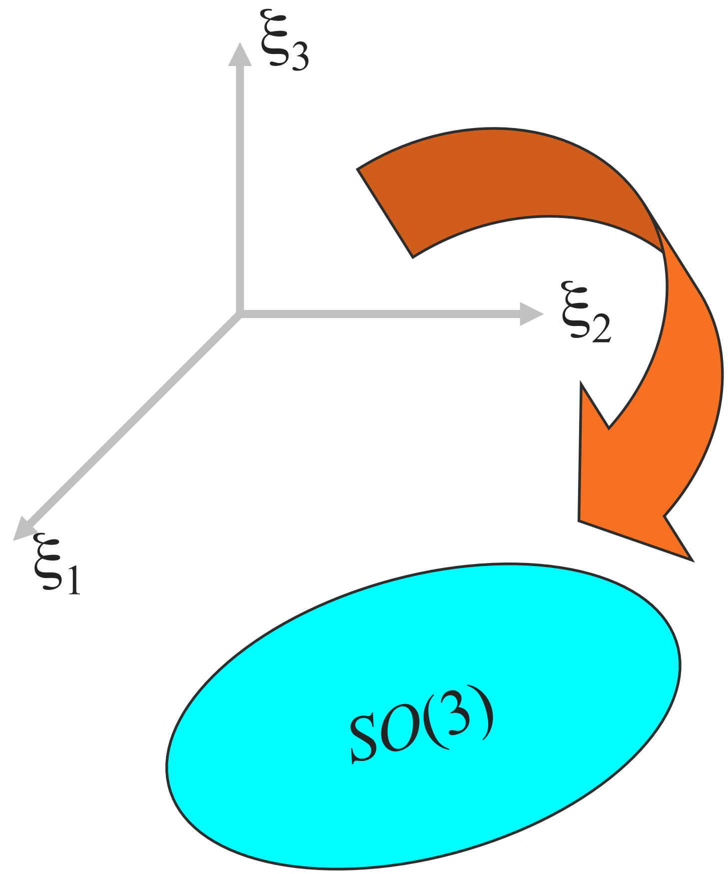
Rotation Matrices

Properties of Rotation Matrices

- Orthogonal
  - Matrix times its transpose equals 1
- Special orthogonal
  - Determinant is +1
- Closed under multiplication
- Composition

  $$^{A}\mathbf{R}_{C} = {}^{A}\mathbf{R}_{B} \times {}^{B}\mathbf{R}_{C}$$

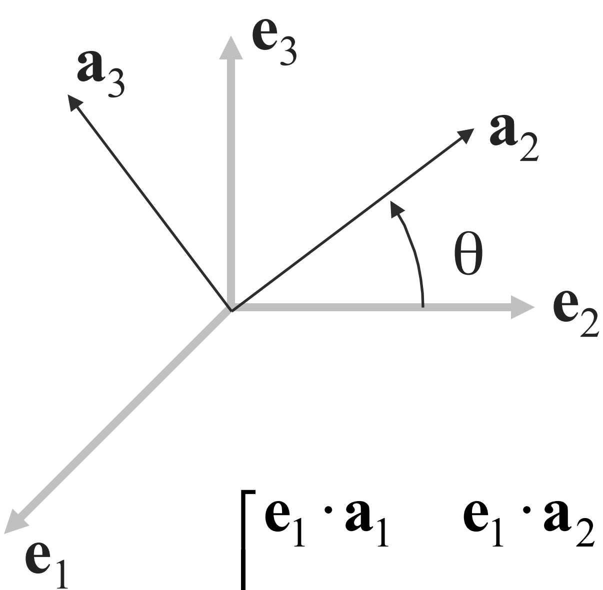
- The inverse of a rotation matrix is also a rotation matrix
- Composition and inverse operations are "continuous functions"
- The set of all rotations is a *Lie Group*, *SO*(3)
- *SO*(3) can be parameterized by 3 coordinates $\boxed{\xi_1 \quad \xi_2 \quad \xi_3}$



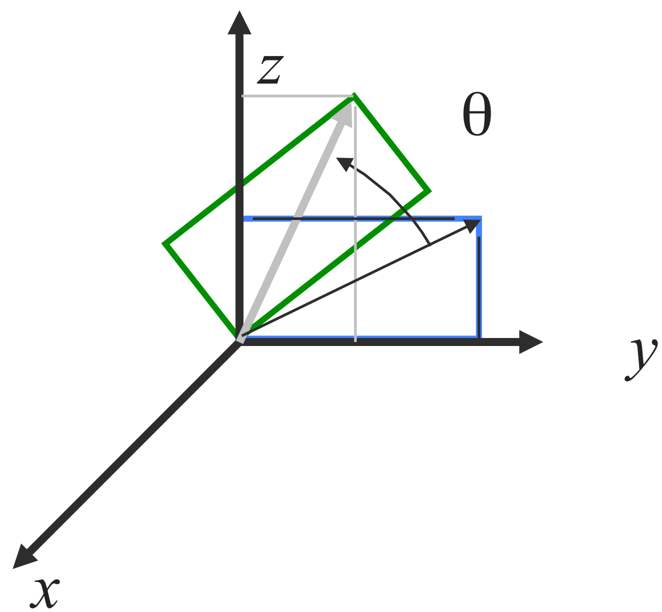
Example: "Simple" Rotations

- Rotation about the *x*-axis through θ

$$Rot(x,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

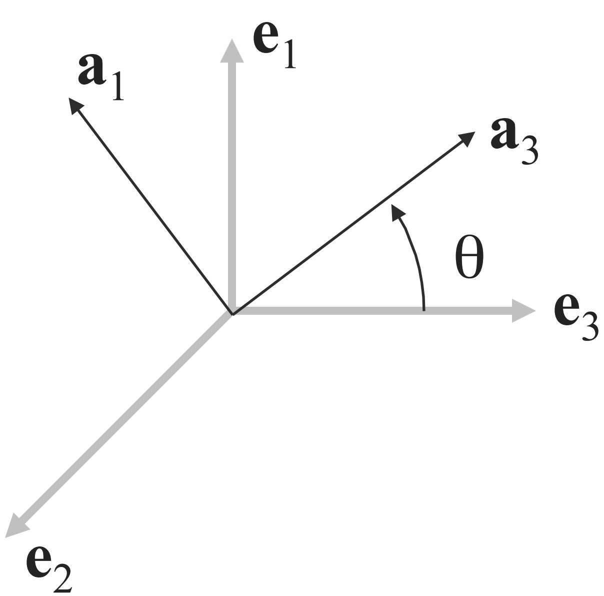


$$\begin{bmatrix} \mathbf{e}_1 \cdot \mathbf{a}_1 & \mathbf{e}_1 \cdot \mathbf{a}_2 & \mathbf{e}_1 \cdot \mathbf{a}_3 \\ \mathbf{e}_2 \cdot \mathbf{a}_1 & \mathbf{e}_2 \cdot \mathbf{a}_2 & \mathbf{e}_2 \cdot \mathbf{a}_3 \\ \mathbf{e}_3 \cdot \mathbf{a}_1 & \mathbf{e}_3 \cdot \mathbf{a}_2 & \mathbf{e}_3 \cdot \mathbf{a}_3 \end{bmatrix}$$



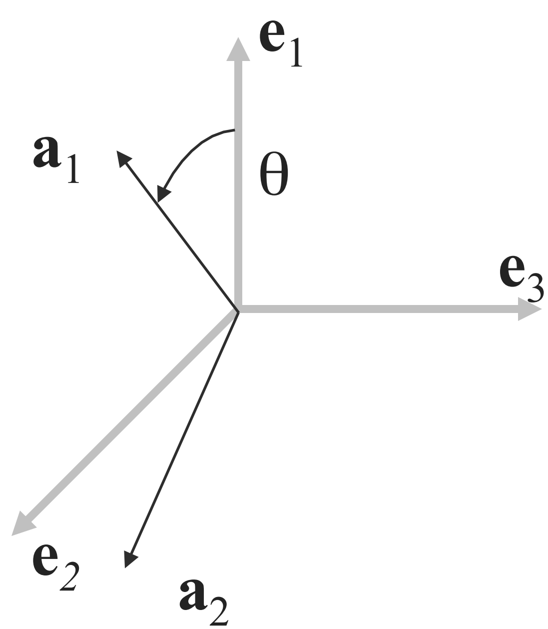
Example: Rotation

Rotation about the *y*-axis through θ



$$\begin{bmatrix} \mathbf{e}_1 \cdot \mathbf{a}_1 & \mathbf{e}_1 \cdot \mathbf{a}_2 & \mathbf{e}_1 \cdot \mathbf{a}_3 \\ \mathbf{e}_2 \cdot \mathbf{a}_1 & \mathbf{e}_2 \cdot \mathbf{a}_2 & \mathbf{e}_2 \cdot \mathbf{a}_3 \\ \mathbf{e}_3 \cdot \mathbf{a}_1 & \mathbf{e}_3 \cdot \mathbf{a}_2 & \mathbf{e}_3 \cdot \mathbf{a}_3 \end{bmatrix}$$

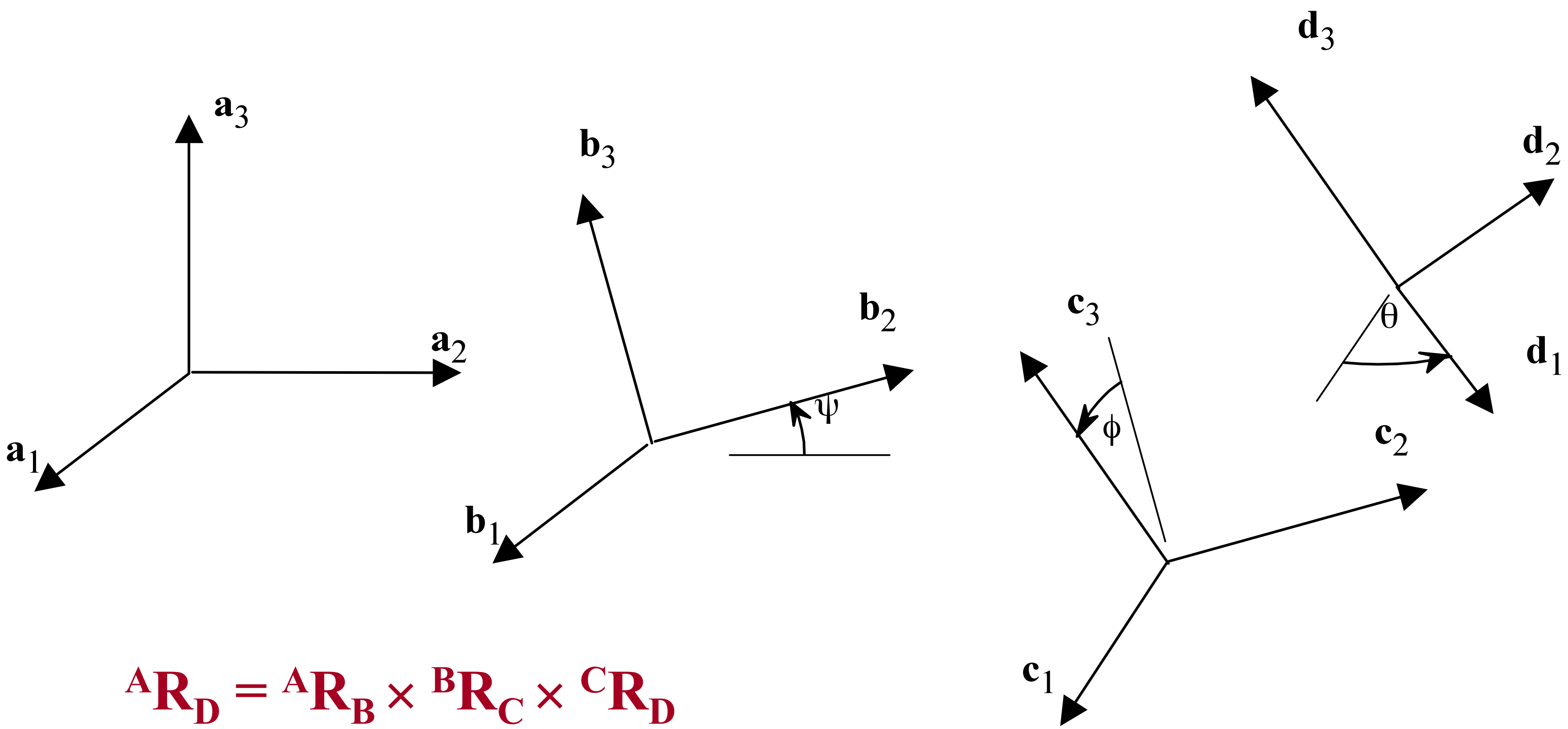
Rotation about the *z*-axis through θ



$$Rot(y,\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$Rot(z,\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Composition of Three Rotations



$$ {}^{A}R_{D} = {}^{A}R_{B} \times {}^{B}R_{C} \times {}^{C}R_{D} $$

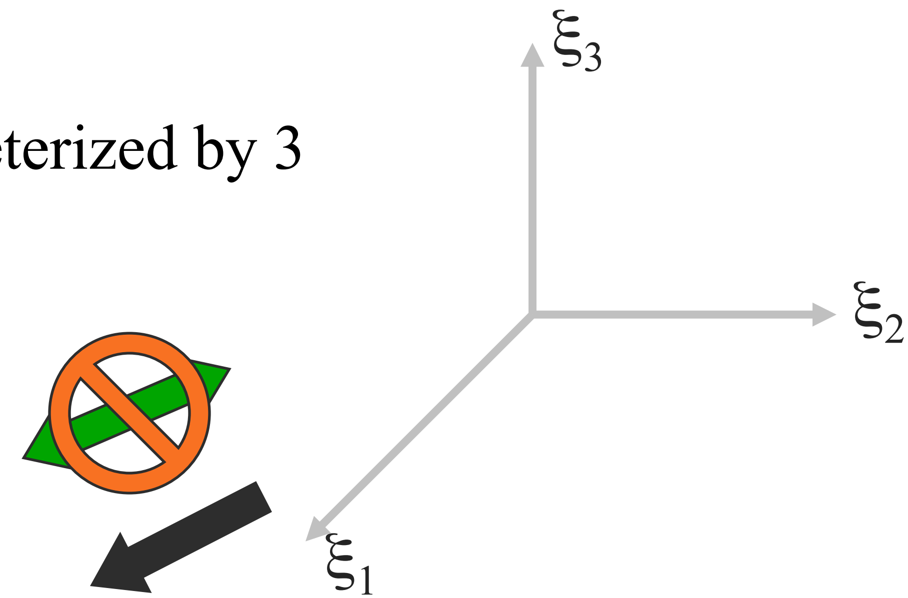
$$ {}^{A}R_{D} = \mathrm{Rot}(x, \psi) \times \mathrm{Rot}(y, \phi) \times \mathrm{Rot}(z, \theta) $$

Euler Angles

Any rotation can be described by three rotations about linearly independent axes.

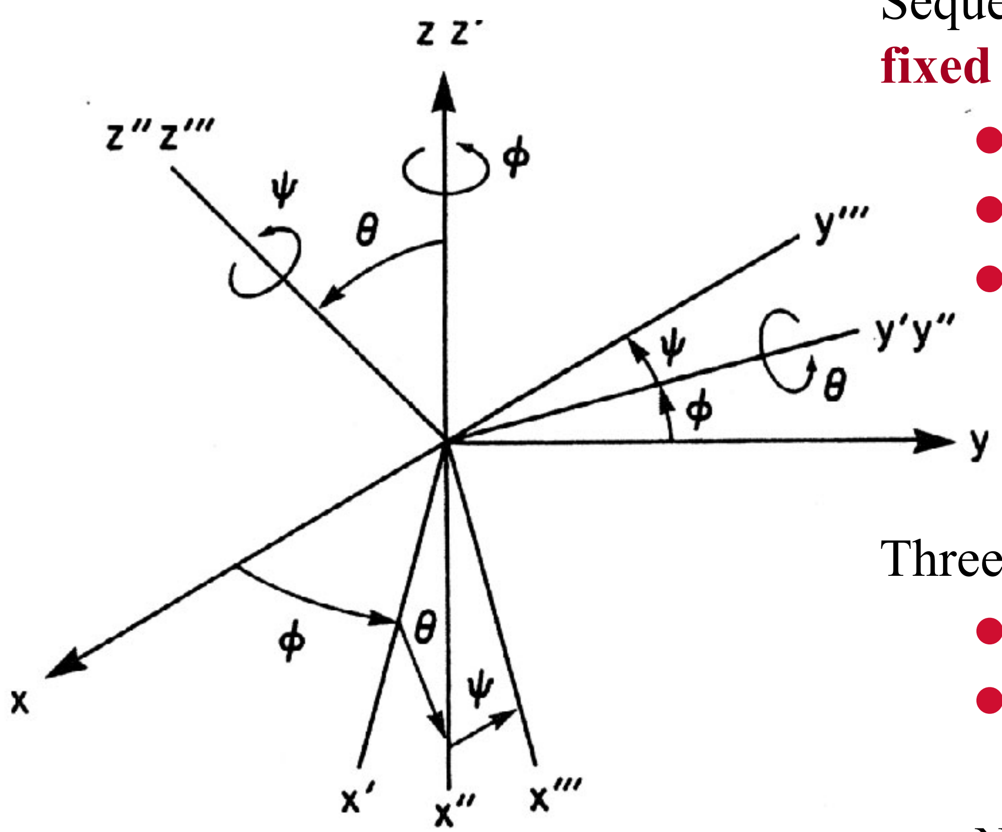
Rotations can be parameterized by 3 coordinates (angles)

$$\left(\text{3} \times \text{3 rotation matrix} \right)$$



Almost 1-1 transformation

Euler Angles: Parameterization of Rotation Matrices



Sequence of three rotations about **body-fixed** axes

- Rot(z, ϕ)
- Rot(y, θ)
- Rot(z, ψ)

Are these LI?

Three Euler Angles

- ϕ, θ, and ψ
- Parameterize rotations

Note

- $\theta = 0$ is a special (singular) case

$$\mathbf{R} = \mathrm{Rot}(z, \phi) \times \mathrm{Rot}(y, \theta) \times \mathrm{Rot}(z, \psi)$$

Determination of Euler Angles

$$\mathbf{R} = \mathrm{Rot}(z, \phi) \times \mathrm{Rot}(y, \theta) \times \mathrm{Rot}(z, \psi)$$

$$R = \begin{bmatrix} \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & -\cos\phi\cos\theta\sin\psi - \sin\phi\cos\psi & \boxed{\cos\phi\sin\theta} \\ \sin\phi\cos\theta\cos\psi + \cos\phi\sin\psi & -\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\sin\theta \\ -\sin\theta\cos\psi & \sin\theta\sin\psi & \boxed{\cos\theta} \end{bmatrix}$$

$$R_{31} = -\sin\theta\cos\psi$$

$$R_{32} = \sin\theta\sin\psi$$

$$\begin{bmatrix} R_{1,1} & R_{1,2} & \boxed{R_{1,3}} \\ R_{2,1} & R_{2,2} & R_{2,3} \\ R_{3,1} & R_{3,2} & \boxed{R_{3,3}} \end{bmatrix}$$

$$R_{33} = \cos\theta$$

$$R_{13} = \sin\theta\cos\phi$$

$$R_{23} = \sin\theta\sin\phi$$

Determination of Euler Angles

If $|R_{33}| < 1$,

$$\theta = \sigma \arccos(R_{33}), \quad \sigma = \pm 1$$

$$\psi = a\tan 2\left(\frac{R_{32}}{\sin\theta}, \frac{-R_{31}}{\sin\theta}\right)$$

$$\phi = a\tan 2\left(\frac{R_{23}}{\sin\theta}, \frac{R_{13}}{\sin\theta}\right)$$

$$R = \begin{bmatrix} \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & -\cos\phi\cos\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\sin\theta \\ \sin\phi\cos\theta\cos\psi + \cos\phi\sin\psi & -\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\sin\theta \\ -\sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta \end{bmatrix}$$

If $R_{33} = 1$,

$$R = \begin{bmatrix} \cos\phi\cos\psi - \sin\phi\sin\psi & -\cos\phi\sin\psi - \sin\phi\cos\psi & 0 \\ \sin\phi\cos\psi + \cos\phi\sin\psi & -\sin\phi\sin\psi + \cos\phi\cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\mathbf{R} = \mathrm{Rot}(z, \phi) \times \mathrm{Rot}(y, \theta) \times \mathrm{Rot}(z, \psi)}$$

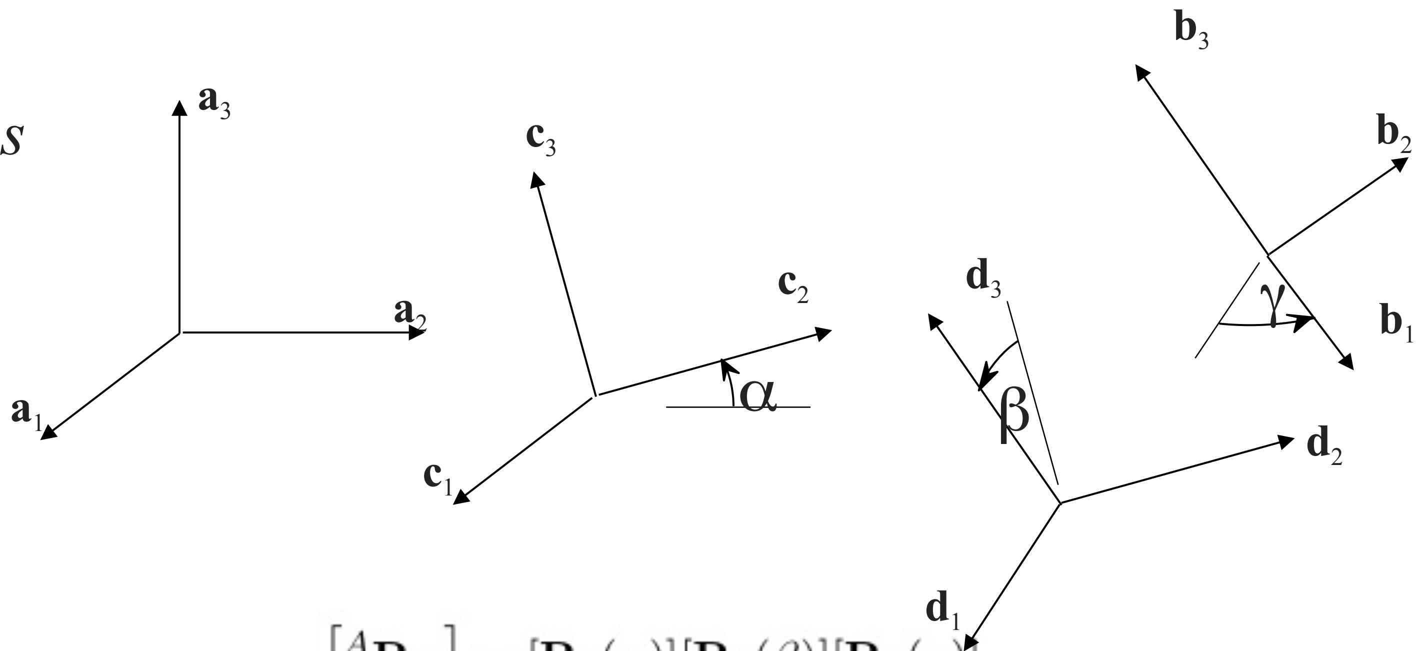
If $R_{33} = -1$,

$$R = \begin{bmatrix} -\cos\phi\cos\psi - \sin\phi\sin\psi & \cos\phi\sin\psi - \sin\phi\cos\psi & 0 \\ \cos\phi\sin\psi - \sin\phi\cos\psi & \sin\phi\sin\psi + \cos\phi\cos\psi & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Two sets of Euler angles for every **R** for almost all **R**'s!

Euler Angles

See notes



$$\left[{}^{A}\mathbf{R}_{B}\right] = [\mathbf{R}_1(\alpha)][\mathbf{R}_2(\beta)][\mathbf{R}_3(\gamma)]$$

$$\left[{}^{A}\mathbf{R}_{B}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix} \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Euler's Theorem

Rotations

Any displacement of a rigid body such that a point on the rigid body, say O, remains fixed, is equivalent to a rotation about a fixed axis through the point O.

Chasles' Theorem for General Displacements (later)

The most general rigid body displacement can be produced by a translation along a line followed (or preceded) by a rotation about that line.

Proof of Euler's Theorem

Displacement from $\{F\}$ to $\{M\}$ $\quad {}^{F}[\mathbf{r}] = {}^{F}\mathbf{R}_{M}\, {}^{M}[\mathbf{r}]$

$$\mathbf{P} = \mathbf{R}\mathbf{p}$$

Solve the eigenvalue problem (find the vector that is unaffected by **R**):

$$\mathbf{R}\mathbf{p} = \lambda\,\mathbf{p}$$

$$|\mathbf{R} - \lambda\mathbf{I}| = 0$$

$$\begin{aligned}&-\lambda^3 + \lambda^2(R_{11} + R_{22} + R_{33})\\ &-\lambda[(R_{22}R_{33} - R_{32}R_{23}) +\\ &(R_{11}R_{33} - R_{13}R_{31}) + (R_{11}R_{22} - R_{12}R_{21})] + 1 = 0\end{aligned} \quad\Rightarrow\quad (\lambda - 1)\left(\lambda^2 - \lambda(R_{11} + R_{22} + R_{33} - 1) + 1\right) = 0$$

Let

$$(R_{11} + R_{22} + R_{33} - 1) = 2\cos\phi$$

$$\cos\phi = \frac{1}{2}(R_{11} + R_{22} + R_{33} - 1)$$

Three eigenvalues and eigenvectors are:

$$\lambda_1 = 1, \qquad \mathbf{p}_1 = \mathbf{u}$$

$$\lambda_2 = e^{i\phi}, \qquad \mathbf{p}_2 = \mathbf{x}$$

$$\lambda_3 = e^{-i\phi}, \qquad \mathbf{p}_2 = \bar{\mathbf{x}}$$

Axis and Angle of Rotation

Rotation Matrix to Axis and Angle

1. Solve

$$\mathbf{Ru} = 1\ \mathbf{u}$$

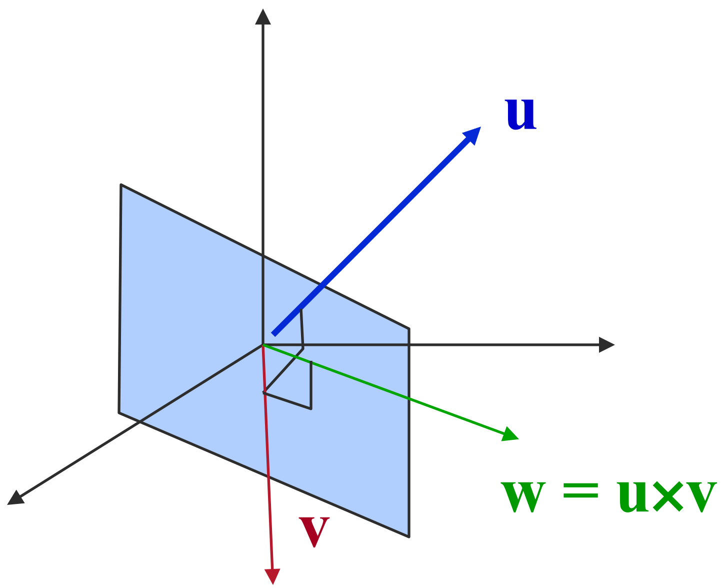
for unit vector along axis

2. Find angle of rotation:

$$\cos\phi = \frac{1}{2}\left(R_{11} + R_{22} + R_{33} - 1\right)$$

Axis and Angle to Rotation Matrix?

The Axis/Angle for a Rotation Matrix



$$\mathbf{Ru} = \mathbf{u}$$

Select **v** to be orthogonal to **u**

Let $\mathbf{w} = \mathbf{u} \times \mathbf{v}$

The other two eigenvectors are perpendicular to **u**

$$\begin{bmatrix} & & \\ & \mathbf{R} & \\ & & \end{bmatrix} \begin{bmatrix} \vdots & \vdots & \vdots \\ \mathbf{v} & \mathbf{w} & \mathbf{u} \\ \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & \vdots \\ \mathbf{v} & \mathbf{w} & \mathbf{u} \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R} = \mathbf{Q}\Lambda\mathbf{Q}^{-1}$$

$$\mathbf{Q}^{-1}\mathbf{RQ} = \Lambda$$

$$\begin{bmatrix} & & \\ & \mathbf{R} & \\ & & \end{bmatrix} \underbrace{\begin{bmatrix} \vdots & \vdots & \vdots \\ \mathbf{v} & \mathbf{w} & \mathbf{u} \\ \vdots & \vdots & \vdots \end{bmatrix}}_{\mathbf{Q}} = \underbrace{\begin{bmatrix} \vdots & \vdots & \vdots \\ \mathbf{v} & \mathbf{w} & \mathbf{u} \\ \vdots & \vdots & \vdots \end{bmatrix}}_{\mathbf{Q}} \underbrace{\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\Lambda}$$

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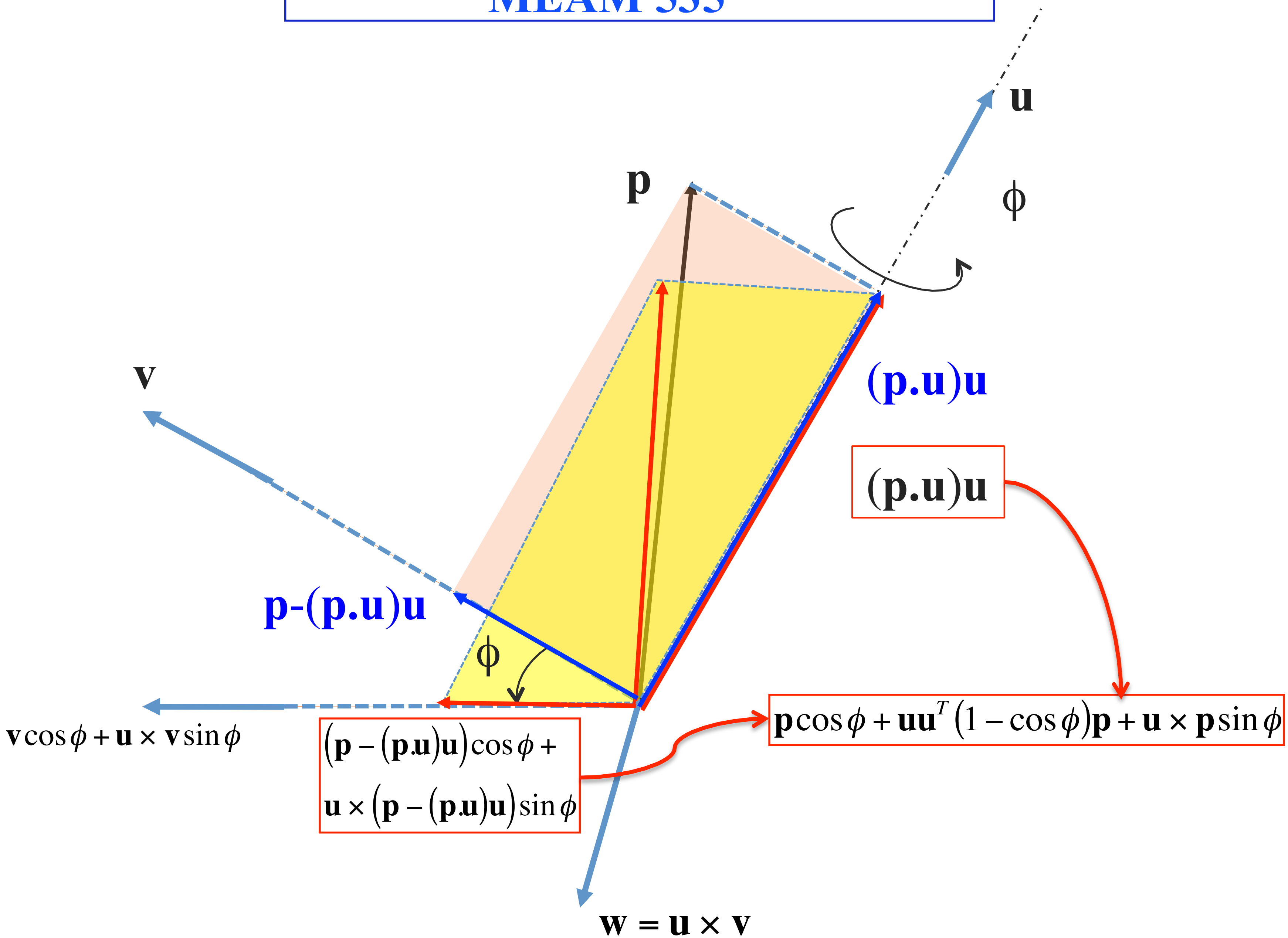
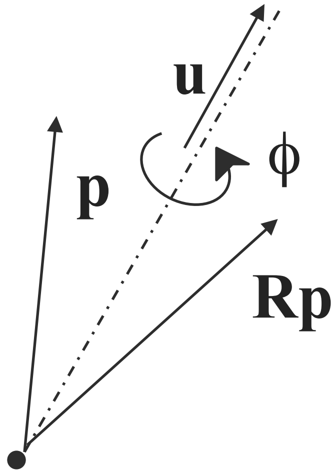


u
ϕ
p
v
(p.u)u
(p.u)u
p-(p.u)u
ϕ
$\mathbf{v}\cos\phi + \mathbf{u}\times\mathbf{v}\sin\phi$
$\mathbf{p}\cos\phi + \mathbf{u}\mathbf{u}^T(1-\cos\phi)\mathbf{p} + \mathbf{u}\times\mathbf{p}\sin\phi$
$(\mathbf{p}-(\mathbf{p.u})\mathbf{u})\cos\phi +$
$\mathbf{u}\times(\mathbf{p}-(\mathbf{p.u})\mathbf{u})\sin\phi$
$\mathbf{w} = \mathbf{u}\times\mathbf{v}$

Axis/Angle to Rotation Matrix

Rotation about **u** through ϕ

$$\mathbf{Rp} = \mathbf{p}\cos\phi + \mathbf{u}\mathbf{u}^T(1-\cos\phi)\mathbf{p} + \mathbf{Up}\sin\phi$$



Rodrigues' formula

$$\boxed{\mathbf{R} = \mathbf{I}\cos\phi + \mathbf{u}\mathbf{u}^T(1-\cos\phi) + \mathbf{U}\sin\phi}$$

where,

$$\mathbf{U} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$$

Notes:

1. Map from **R** to (**u**, ϕ) is one to many.
   - restrict ϕ to the interval $[0,\pi]$
2. Singular
   - $\mathbf{R} = \mathbf{I}$
   - trace(**R**) = -1

Extracting the axis and the angle from the rotation matrix

1. Find the eigenvector corresponding to λ=1.
2. From Rodrigues' formula:

$$\cos\phi = \frac{1}{2}\left(R_{11} + R_{22} + R_{33} - 1\right)$$

$$\mathbf{U} = \frac{1}{2\sin\phi}\left(\mathbf{R} - \mathbf{R}^T\right)$$

See notes - rotation operator $\mathcal{R}_{\hat{\mathbf{n}}}(\varphi)$

The 3×1 vector **a** and its 3×3 skew symmetric matrix counterpart **A**

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ -a_2 b_1 + a_1 b_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

a

A

For any vector b

$$\mathbf{a} \times \mathbf{b} = \mathbf{A}\,\mathbf{b}$$