Scalar and Vector Functions of a Single Scalar Variable

C.1 Introduction and Purpose

In this appendix we shall investigate vector functions of a single scalar variable. For dynamics purposes, this single scalar variable can be thought of as time and will be denoted by \( t \). This study will be fundamental to the study of the kinematics of particles and of rigid bodies.

C.2 Scalar Functions of a Scalar Variable

Recall that when we refer to a scalar in dynamics, we are really referring to a real number so that the set of all scalars is just the set of all real numbers and we will denote this set by the symbol \( R \). A scalar function of a scalar variable is simply a rule that maps a subset of \( R \) (called the domain and denoted by \( A \)) to a subset of \( R \) (called the co-domain and denoted by \( B \)). Such a function is denoted by the symbolism, \( f : A \to B \). The value of \( f \) at some scalar \( t \) in \( A \) is denoted by \( f(t) \) and it represents the scalar (in \( B \)) that the scalar \( t \) (in \( A \)) is mapped into by the function \( f \). This is represented symbolically as follows.

\[
(t \in A) \rightarrow f \rightarrow (f(t) \in B).
\]
Most books use the notation \( f : \mathbb{R} \to \mathbb{R} \) even when \( A \neq \mathbb{R} \) and when \( B \neq \mathbb{R} \). Since we are not interested in the precise details regarding functional analysis, we shall adopt the notation \( f : \mathbb{R} \to \mathbb{R} \) as well, as long as it causes no confusion.

It is assumed that the student is already familiar with scalar functions of a scalar variable since these are just the functions encountered in basic calculus. Given a function \( f : \mathbb{R} \to \mathbb{R} \), the derivative function (if it exist) is defined by the function \( \dot{f} : \mathbb{R} \to \mathbb{R} \) where

\[
\dot{f}(t) \equiv \lim_{\Delta t \to 0} \left( \frac{f(t + \Delta t) - f(t)}{\Delta t} \right)
\]  
(C.1a)

Of course, this is also denoted by

\[
\frac{df(t)}{dt} \equiv \lim_{\Delta t \to 0} \left( \frac{f(t + \Delta t) - f(t)}{\Delta t} \right),
\]  
(C.1b)

or \( f'(t) \). Throughout this appendix, there is only a single reference frame \( S \) in which all measurements are being made. Consequently, we need not use the more detailed notation

\[
\frac{sdf(t)}{dt} \equiv \lim_{\Delta t \to 0} \left( \frac{f(t + \Delta t) - f(t)}{\Delta t} \right),
\]

when referring to the derivative of \( f \) with respect to \( t \) as measured in \( S \).

In addition, we shall reserve the ”dot” notation to mean differentiation with respect to the variable \( t \) and we shall reserve the ”prime” notation to be differentiation with respect to the argument of the function, and in general, these are not the same when the argument of \( f \) is not \( t \), but some function of \( t \). For example, suppose that \( f : \mathbb{R} \to \mathbb{R} \) such that \( f(t) = t^2 \). Then \( f(t^2) = t^4 \) and

\[
\dot{f}(t^2) = \frac{df(t^2)}{dt} = \frac{dt^4}{dt} = 4t^3
\]
while

\[
f'(t^2) = \frac{df(t^2)}{dt^2} = \frac{dt^4}{dt^2} = \frac{4t^3}{2t} = 2t^2.
\]

This shows that, in general, \( \dot{f}(t^2) \neq f'(t^2) \) so one must be very careful about the notation. Of course, we will always have

\[
\dot{f}(t) = f'(t).
\]
A function $f$ for which $\dot{f}$ exists is called a *differentiable scalar function* of the scalar variable $t$. Some very basic functions and their corresponding derivatives are already familiar to the student and are summarized in the following table.

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$\dot{f}(t)$</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t^\alpha$</td>
<td>$\alpha t^{\alpha-1}$</td>
<td>$\alpha = \text{real number}$</td>
</tr>
<tr>
<td>$e^{\alpha t}$</td>
<td>$\alpha e^{\alpha t}$</td>
<td>$\alpha = \text{real number}$</td>
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<tr>
<td>$\sin(\alpha t)$</td>
<td>$\alpha \cos(\alpha t)$</td>
<td>$\alpha = \text{real number}$</td>
</tr>
<tr>
<td>$\cos(\alpha t)$</td>
<td>$-\alpha \sin(\alpha t)$</td>
<td>$\alpha = \text{real number}$</td>
</tr>
<tr>
<td>$\sinh(\alpha t)$</td>
<td>$\alpha \cosh(\alpha t)$</td>
<td>$\alpha = \text{real number}$</td>
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<tr>
<td>$\cosh(\alpha t)$</td>
<td>$\alpha \sinh(\alpha t)$</td>
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<tr>
<td>$\ln</td>
<td>t</td>
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</tr>
<tr>
<td>$\alpha^t$</td>
<td>$\alpha^t \ln(\alpha)$</td>
<td>$\alpha = \text{positive real number}$</td>
</tr>
</tbody>
</table>

Some basic properties of differentiation should already be familiar to the student and these are summarized as follows:

*Properties of the Derivative: Scalar Functions*

Let $f$ and $g$ be differentiable scalar functions of a scalar variable $t$, then each of the following is true.

a.) Constant Rule:

$$\frac{df(t)}{dt} = 0 \quad \text{if and only if} \quad f(t) = \text{a constant} \quad (C.2a)$$

for all $t$ in the domain of $f$.

b.) Sum and Difference Rule:

$$\frac{d(f(t) \pm g(t))}{dt} = \frac{df(t)}{dt} \pm \frac{dg(t)}{dt}. \quad (C.2b)$$

c.) Constant Product Rule:

$$\frac{d(\alpha f(t))}{dt} = \alpha \frac{df(t)}{dt} \quad (C.2c)$$

for any scalar constant $\alpha$.  

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d.) Variable Product Rule:
\[
\frac{d(f(t)g(t))}{dt} = \frac{df(t)}{dt}g(t) + f(t)\frac{dg(t)}{dt}.
\]  
(C.2d)

e.) Quotient Rule:
\[
\frac{d(f(t)/g(t))}{dt} = \frac{(df(t)/dt)g(t) - f(t)(dg(t)/dt)}{(g(t))^2}.
\]  
(C.2e)

f.) Chain Rule:
\[
\frac{d(f(g(t)))}{dt} = \frac{df(g(t))}{dg(t)}\frac{dg(t)}{dt} = f'(g(t))g(t)
\]  
(C.2f)

which is not the same as \(\dot{f}(g(t))\dot{g}(t)\).

The proofs of these are very basic and the student should already be familiar with most of these.

C.3 Vector Functions of a Scalar Variable

A vector function of a scalar variable is simply a rule that maps a subset of \(R\) to a subset of \(G\) (the set of all vectors in space). Such a function is denoted by the symbolism, \(\textbf{F} : R \rightarrow G\). The value of \(\textbf{F}\) at some scalar \(t\) in \(R\) is denoted by \(\textbf{F}(t)\) and it represents the vector (in \(G\)) that the scalar \(t\) (in \(R\)) is mapped into by the function \(\textbf{F}\). This is represented symbolically as follows.

\[
(t \in R) \rightarrow \textbf{F} \rightarrow (\textbf{F}(t) \in G).
\]

Given a function \(\textbf{F} : R \rightarrow G\), the derivative function (if it exist) is defined by \(\dot{\textbf{F}} : R \rightarrow G\) where
\[
\dot{\textbf{F}}(t) \equiv \lim_{\Delta t \rightarrow 0} \left( \frac{\textbf{F}(t + \Delta t) - \textbf{F}(t)}{\Delta t} \right)
\]  
(C.3a)

Of course, this is also denoted by
\[
\frac{d\textbf{F}(t)}{dt} \equiv \lim_{\Delta t \rightarrow 0} \left( \frac{\textbf{F}(t + \Delta t) - \textbf{F}(t)}{\Delta t} \right),
\]  
(C.3b)

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or $\mathbf{F}'(t)$. Once again, we point out that throughout this appendix, there is only a single reference frame $S$ in which all measurements are being made. Consequently, we need not use the more detailed notation

$$
\frac{Sd\mathbf{F}(t)}{dt} = \lim_{\Delta t \to 0} \left( \frac{\mathbf{F}(t + \Delta t) - \mathbf{F}(t)}{\Delta t} \right),
$$

when referring to the derivative of $\mathbf{F}$ with respect to $t$ as measured in $S$.

As with scalar functions above, we shall reserve the "dot" notation to mean differentiation with respect to the variable $t$ and we shall reserve the "prime" notation to be differentiation with respect to the argument of the function, and in general, these are not the same when the argument of $\mathbf{F}$ is not $t$, but some function of $t$.

A function $\mathbf{F}$ for which $\dot{\mathbf{F}}$ exists is called a differentiable vector function of the variable $t$. Some basic properties of differentiation are summarized in the following properties.

**Properties of the Derivative: Vector Functions**

Let $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ be differentiable vector functions of a variable $t$, then each of the following is true.

a.) Constant Rule:

$$
\frac{d}{dt}\mathbf{F}(t) = \mathbf{0} \quad \text{if and only if} \quad \mathbf{F}(t) = \mathbf{C} \quad (C.4a)
$$

an constant vector.

b.) Sum and Difference Rule:

$$
\frac{d}{dt} (\mathbf{F}(t) \pm \mathbf{G}(t)) = \frac{d\mathbf{F}(t)}{dt} \pm \frac{d\mathbf{G}(t)}{dt} \quad (C.4b)
$$

c.) Scalar-Product Rule: For any differentiable scalar function $\alpha(t)$, we have

$$
\frac{d}{dt} (\alpha(t)\mathbf{F}(t)) = \frac{d\alpha(t)}{dt}\mathbf{F}(t) + \alpha(t)\frac{d\mathbf{F}(t)}{dt} \quad (C.4c)
$$
d.) Dot-Product Rule:

\[
\frac{d}{dt} (F(t) \cdot G(t)) = \frac{dF(t)}{dt} \cdot G(t) + F(t) \cdot \frac{dG(t)}{dt}
\]  \hspace{1cm} (C.4d)

e.) Cross-Product Rule:

\[
\frac{d}{dt} (F(t) \times G(t)) = \frac{dF(t)}{dt} \times G(t) + F(t) \times \frac{dG(t)}{dt}
\]  \hspace{1cm} (C.4e)

f.) Tensor-Product (Dyadic) Rule:

\[
\frac{d}{dt} (F(t)G(t)) = \frac{dF(t)}{dt}G(t) + F(t)\frac{dG(t)}{dt}
\]  \hspace{1cm} (C.4f)

g.) Triple-Scalar-Product Rule:

\[
\frac{d}{dt}[F(t), G(t), H(t)] = \left[ \frac{dF(t)}{dt}, G(t), H(t) \right] + \left[ F(t), \frac{dG(t)}{dt}, H(t) \right] + \left[ F(t), G(t), \frac{dH(t)}{dt} \right]
\]  \hspace{1cm} (C.4g)

h.) Magnitude Rule:

\[
|F(t)|\frac{d|F(t)|}{dt} = F(t) \cdot \frac{dF(t)}{dt}
\]  \hspace{1cm} (C.4h)

Note that the magnitude rule follows from the dot-product rule with $G = F$ since we have

\[
\frac{d}{dt} (F(t) \cdot F(t)) = \frac{dF(t)}{dt} \cdot F(t) + F(t) \cdot \frac{dF(t)}{dt}
\]

or

\[
\frac{d}{dt}|F(t)|^2 = F(t) \cdot \frac{dF(t)}{dt} + F(t) \cdot \frac{dF(t)}{dt}
\]

or

\[
2|F(t)|\frac{d|F(t)|}{dt} = 2F(t) \cdot \frac{dF(t)}{dt}
\]

which reduces to Equation (C.4h).
i.) Constant Magnitude Rule: If $|F(t)| = \text{constant}$, then

\[ \frac{dF(t)}{dt} \perp F(t). \tag{C.4i} \]

This follows from Equation (C.4h) since for $|F(t)| = \text{constant}$, we get

\[ |F(t)| \frac{d}{dt}|F(t)| = F(t) \cdot \frac{dF(t)}{dt} = 0. \]

Note that for any unit vector, which has a constant magnitude of 1, then,

\[ \frac{d\hat{u}(t)}{dt} \perp \hat{u}(t). \]

j.) Chain Rule: For any differentiable scalar function $\alpha(t)$, we have

\[ \frac{dF(\alpha(t))}{dt} = \frac{dF(\alpha(t))}{d\alpha(t)} \frac{d\alpha(t)}{dt} = F'(\alpha(t))\dot{\alpha}(t) \tag{C.4j} \]

which is not the same as $\dot{F}(\alpha(t))\dot{\alpha}(t)$.

Once again, the proofs of these are very basic and the student should already be familiar with most of these.

**C.4 Vector Representations and Differentiation: General Approach**

Suppose that $S\{t\} = \{\hat{e}_1(t), \hat{e}_2(t), \hat{e}_3(t)\}$ is a standard-reference triad (SRT), which can change with the scalar variable $t$. This means that although $\{\hat{e}_1(t), \hat{e}_2(t), \hat{e}_3(t)\}$ are changing with the scalar variable $t$, it remains that

\[ \hat{e}_1(t) \times \hat{e}_2(t) = \hat{e}_3(t) \quad \text{and} \quad \hat{e}_2(t) \times \hat{e}_3(t) = \hat{e}_1(t) \]

and

\[ \hat{e}_3(t) \times \hat{e}_1(t) = \hat{e}_2(t) \]

for all values of the scalar variable $t$. For example, $S\{t\} = \{\hat{e}_1(t), \hat{e}_2(t), \hat{e}_3(t)\}$ with

\[ \hat{e}_1(t) = \sin(2t)\cos(t)\hat{e}_x + \sin(2t)\sin(t)\hat{e}_y + \cos(2t)\hat{e}_z \]

and

\[ \hat{e}_2(t) = \cos(2t)\cos(t)\hat{e}_x + \cos(2t)\sin(t)\hat{e}_y + \sin(2t)\hat{e}_z \]

and

\[ \hat{e}_3(t) = \sin(t)\hat{e}_x + \cos(t)\hat{e}_y \]
and
\[ \hat{\mathbf{e}}_2(t) = \cos(2t)\cos(t)\hat{\mathbf{e}}_x + \cos(2t)\sin(t)\hat{\mathbf{e}}_y - \sin(2t)\hat{\mathbf{e}}_z \]
and
\[ \hat{\mathbf{e}}_3(t) = -\sin(t)\hat{\mathbf{e}}_x + \cos(t)\hat{\mathbf{e}}_y \]
is a SRT that changes with the variable \( t \), and the student should show that
\[ \hat{\mathbf{e}}_1(t) \times \hat{\mathbf{e}}_2(t) = \hat{\mathbf{e}}_3(t) \quad , \quad \hat{\mathbf{e}}_2(t) \times \hat{\mathbf{e}}_3(t) = \hat{\mathbf{e}}_1(t) \]
and
\[ \hat{\mathbf{e}}_3(t) \times \hat{\mathbf{e}}_1(t) = \hat{\mathbf{e}}_2(t) \]
for all values of \( t \). Even through \( S\{t\} \) changes with \( t \), we shall represent it simply as \( S \), without there being any confusion.

**C.4a First Derivatives**

Suppose that a vector function \( \mathbf{F}(t) \) is represented in terms of \( S \) as
\[ \mathbf{F}(t) = F_1(t)\hat{\mathbf{e}}_1(t) + F_2(t)\hat{\mathbf{e}}_2(t) + F_3(t)\hat{\mathbf{e}}_3(t) \]
then, using the properties of differentiation of vector functions, we have
\[
\frac{d\mathbf{F}(t)}{dt} = \frac{d}{dt}(F_1(t)\hat{\mathbf{e}}_1(t) + F_2(t)\hat{\mathbf{e}}_2(t) + F_3(t)\hat{\mathbf{e}}_3(t))
\]
\[
= \frac{d}{dt}F_1(t)\hat{\mathbf{e}}_1(t) + F_1(t)\frac{d}{dt}\hat{\mathbf{e}}_1(t)
\]
\[
+ \frac{d}{dt}F_2(t)\hat{\mathbf{e}}_2(t) + F_2(t)\frac{d}{dt}\hat{\mathbf{e}}_2(t)
\]
\[
+ \frac{d}{dt}F_3(t)\hat{\mathbf{e}}_3(t) + F_3(t)\frac{d}{dt}\hat{\mathbf{e}}_3(t)
\]
or simply
\[
\frac{d\mathbf{F}(t)}{dt} = \frac{dF_1(t)}{dt}\hat{\mathbf{e}}_1(t) + \frac{dF_2(t)}{dt}\hat{\mathbf{e}}_2(t) + \frac{dF_3(t)}{dt}\hat{\mathbf{e}}_3(t)
\]
\[
+ F_1(t)\frac{d\hat{\mathbf{e}}_1(t)}{dt} + F_2(t)\frac{d\hat{\mathbf{e}}_2(t)}{dt} + F_3(t)\frac{d\hat{\mathbf{e}}_3(t)}{dt}. \quad (C.5)
\]

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Now we know that
\[
\frac{d\hat{e}_1(t)}{dt}, \quad \frac{d\hat{e}_2(t)}{dt}, \quad \frac{d\hat{e}_3(t)}{dt}
\]
are all vectors and so we may expand each of these in terms of the SRT \( S \) and write
\[
\frac{d\hat{e}_1(t)}{dt} = a_{11}(t)\hat{e}_1(t) + a_{12}(t)\hat{e}_2(t) + a_{13}(t)\hat{e}_3(t) \quad \text{(C.6a)}
\]
and
\[
\frac{d\hat{e}_2(t)}{dt} = a_{21}(t)\hat{e}_1(t) + a_{22}(t)\hat{e}_2(t) + a_{23}(t)\hat{e}_3(t) \quad \text{(C.6b)}
\]
and
\[
\frac{d\hat{e}_3(t)}{dt} = a_{31}(t)\hat{e}_1(t) + a_{32}(t)\hat{e}_2(t) + a_{33}(t)\hat{e}_3(t). \quad \text{(C.6c)}
\]
But we also know (since each vector in the SRT has constant magnitude of 1) that
\[
\frac{d\hat{e}_1(t)}{dt} \perp \hat{e}_1(t), \quad \frac{d\hat{e}_2(t)}{dt} \perp \hat{e}_2(t), \quad \frac{d\hat{e}_3(t)}{dt} \perp \hat{e}_3(t)
\]
and so we must have
\[
\frac{d\hat{e}_1(t)}{dt} \cdot \hat{e}_1(t) = (a_{11}(t)\hat{e}_1(t) + a_{12}(t)\hat{e}_2(t) + a_{13}(t)\hat{e}_3(t)) \cdot \hat{e}_1(t) = a_{11}(t)(\hat{e}_1(t) \cdot \hat{e}_1(t)) + a_{12}(t)(\hat{e}_2(t) \cdot \hat{e}_1(t)) + a_{13}(t)(\hat{e}_3(t) \cdot \hat{e}_1(t)) = a_{11}(t)(1) + a_{12}(t)(0) + a_{13}(t)(0) = a_{11}(t) = 0.
\]
In a similar way we must have
\[
\frac{d\hat{e}_2(t)}{dt} \cdot \hat{e}_2(t) = a_{22}(t) = 0 \quad \text{and} \quad \frac{d\hat{e}_3(t)}{dt} \cdot \hat{e}_3(t) = a_{33}(t) = 0.
\]
Thus Equations (C.6a,b,c) for the derivatives of the unit vectors in the SRT reduce to
\[
\frac{d\hat{e}_1(t)}{dt} = a_{12}(t)\hat{e}_2(t) + a_{13}(t)\hat{e}_3(t) \quad \text{(C.7a)}
\]
and
\[
\frac{d\hat{e}_2(t)}{dt} = a_{21}(t)\hat{e}_1(t) + a_{23}(t)\hat{e}_3(t) \quad \text{(C.7b)}
\]
and

\[ \frac{d \hat{e}_3(t)}{dt} = a_{31}(t)\hat{e}_1(t) + a_{32}(t)\hat{e}_2(t). \]  \hspace{1cm} \text{(C.7c)}

Now, since \( S \) is a SRT, we also know that

\[ \hat{e}_1(t) \times \hat{e}_2(t) = \hat{e}_3(t) \quad \text{and} \quad \hat{e}_2(t) \times \hat{e}_3(t) = \hat{e}_1(t) \]

and

\[ \hat{e}_3(t) \times \hat{e}_1(t) = \hat{e}_2(t) \]

and so, for example,

\[
\frac{d \hat{e}_3(t)}{dt} = \frac{d(\hat{e}_1(t) \times \hat{e}_2(t))}{dt} = \frac{d\hat{e}_1(t)}{dt} \times \hat{e}_2(t) + \hat{e}_1(t) \times \frac{d\hat{e}_2(t)}{dt}
\]

or

\[
\frac{d \hat{e}_3(t)}{dt} = (a_{12}(t)\hat{e}_2(t) + a_{13}(t)\hat{e}_3(t)) \times \hat{e}_2(t)
\]

\[
+ \hat{e}_1(t) \times (a_{21}(t)\hat{e}_1(t) + a_{23}(t)\hat{e}_3(t))
\]

\[
= a_{12}(t)(\hat{e}_2(t) \times \hat{e}_2(t)) + a_{13}(t)(\hat{e}_3(t) \times \hat{e}_2(t))
\]

\[
+ a_{21}(t)(\hat{e}_1(t) \times \hat{e}_1(t)) + a_{23}(t)(\hat{e}_1(t) \times \hat{e}_3(t))
\]

\[
= a_{12}(t)(0) + a_{13}(t)(-\hat{e}_1(t)) + a_{21}(t)(0) + a_{23}(t)(-\hat{e}_2(t))
\]

since

\[ \hat{e}_3(t) \times \hat{e}_2(t) = -\hat{e}_1(t) \quad \text{and} \quad \hat{e}_1(t) \times \hat{e}_3(t) = -\hat{e}_2(t) \]

and of course

\[ \hat{e}_1(t) \times \hat{e}_1(t) = \hat{e}_2(t) \times \hat{e}_2(t) = 0. \]

Thus we have

\[
\frac{d \hat{e}_3(t)}{dt} = -a_{13}(t)\hat{e}_1(t) - a_{23}(t)\hat{e}_2(t)
\]

and setting this equal to

\[
\frac{d \hat{e}_3(t)}{dt} = a_{31}(t)\hat{e}_1(t) + a_{32}(t)\hat{e}_2(t)
\]

then leads to \(-a_{13}(t) = a_{31}(t)\) and \(-a_{23}(t) = a_{32}(t)\). By looking at \(d \hat{e}_3(t)/dt\), we find that \(-a_{21}(t) = a_{12}(t)\). Thus Equations (C.7a,b,c) reduce to

\[
\frac{d \hat{e}_1(t)}{dt} = a_{12}(t)\hat{e}_2(t) + a_{13}(t)\hat{e}_3(t) \quad \text{(C.8a)}
\]
and
\[
\frac{d\mathbf{e}_2(t)}{dt} = -a_{12}(t)\mathbf{e}_1(t) + a_{23}(t)\mathbf{e}_3(t) \tag{C.8b}
\]
and
\[
\frac{d\mathbf{e}_3(t)}{dt} = -a_{13}(t)\mathbf{e}_1(t) - a_{23}(t)\mathbf{e}_2(t). \tag{C.8c}
\]

If we define a vector
\[
\mathbf{\omega}(t) = \omega_1(t)\mathbf{e}_1(t) + \omega_2(t)\mathbf{e}_2(t) + \omega_3(t)\mathbf{e}_3(t) \tag{C.9a}
\]
with
\[
\omega_1(t) = a_{23}(t), \quad \omega_2(t) = -a_{13}(t), \quad \omega_3(t) = a_{12}(t) \tag{C.9b}
\]
then Equations (C.8a,b,c) read
\[
\frac{d\mathbf{e}_1(t)}{dt} = \omega_3(t)\mathbf{e}_2(t) - \omega_2(t)\mathbf{e}_3(t) \tag{C.10a}
\]
and
\[
\frac{d\mathbf{e}_2(t)}{dt} = -\omega_3(t)\mathbf{e}_1(t) + \omega_1(t)\mathbf{e}_3(t) \tag{C.10b}
\]
and
\[
\frac{d\mathbf{e}_3(t)}{dt} = \omega_2(t)\mathbf{e}_1(t) - \omega_1(t)\mathbf{e}_2(t). \tag{C.10c}
\]

Using the cross product and the equations
\[
\mathbf{\hat{e}}_1(t) \times \mathbf{\hat{e}}_2(t) = \mathbf{\hat{e}}_3(t), \quad \mathbf{\hat{e}}_2(t) \times \mathbf{\hat{e}}_3(t) = \mathbf{\hat{e}}_1(t)
\]
and
\[
\mathbf{\hat{e}}_3(t) \times \mathbf{\hat{e}}_1(t) = \mathbf{\hat{e}}_2(t),
\]
along with
\[
\mathbf{\hat{e}}_1(t) \times \mathbf{\hat{e}}_1(t) = \mathbf{\hat{e}}_2(t) \times \mathbf{\hat{e}}_2(t) = \mathbf{\hat{e}}_3(t) \times \mathbf{\hat{e}}_3(t) = \mathbf{0},
\]
it is easy to see that Equations (C.10a,b,c) are the same as
\[
\frac{d\mathbf{\hat{e}}_1(t)}{dt} = (\omega_1(t)\mathbf{\hat{e}}_1(t) + \omega_2(t)\mathbf{\hat{e}}_2(t) + \omega_3(t)\mathbf{\hat{e}}_3(t)) \times \mathbf{\hat{e}}_1(t)
\]
and
\[
\frac{d\mathbf{\hat{e}}_2(t)}{dt} = (\omega_1(t)\mathbf{\hat{e}}_1(t) + \omega_2(t)\mathbf{\hat{e}}_2(t) + \omega_3(t)\mathbf{\hat{e}}_3(t)) \times \mathbf{\hat{e}}_2(t)
\]
and
\[ \frac{d\mathbf{e}_3(t)}{dt} = (\omega_1(t)\hat{e}_1(t) + \omega_2(t)\hat{e}_2(t) + \omega_3(t)\hat{e}_3(t)) \times \hat{e}_3(t) \]
which says that
\[ \frac{d\mathbf{e}_k(t)}{dt} = \omega(t) \times \hat{e}_k(t) \quad \text{for} \quad k = 1, 2, 3. \] (C.11)

Note that from Equations (C.10a,b,c), we see that
\[ \omega_1(t) = \frac{d\mathbf{e}_2(t)}{dt} \cdot \hat{e}_3(t) = -\frac{d\mathbf{e}_3(t)}{dt} \cdot \hat{e}_2(t) \] (C.12a)
and
\[ \omega_2(t) = \frac{d\mathbf{e}_3(t)}{dt} \cdot \hat{e}_1(t) = -\frac{d\mathbf{e}_1(t)}{dt} \cdot \hat{e}_3(t) \] (C.12b)
and
\[ \omega_3(t) = \frac{d\mathbf{e}_1(t)}{dt} \cdot \hat{e}_2(t) = -\frac{d\mathbf{e}_2(t)}{dt} \cdot \hat{e}_1(t) \] (C.12c)
and so Equation (C.9a) reads
\[
\omega(t) = \left( \frac{d\mathbf{e}_2(t)}{dt} \cdot \hat{e}_3(t) \right) \hat{e}_1(t) + \left( \frac{d\mathbf{e}_3(t)}{dt} \cdot \hat{e}_1(t) \right) \hat{e}_2(t) \\
+ \left( \frac{d\mathbf{e}_1(t)}{dt} \cdot \hat{e}_2(t) \right) \hat{e}_3(t).
\]
or (with the t dependance suppressed),
\[
\omega = \left( \frac{d\mathbf{e}_2}{dt} \cdot \mathbf{e}_3 \right) \mathbf{e}_1 + \left( \frac{d\mathbf{e}_3}{dt} \cdot \mathbf{e}_1 \right) \mathbf{e}_2 + \left( \frac{d\mathbf{e}_1}{dt} \cdot \mathbf{e}_2 \right) \mathbf{e}_3. \] (C.13a)

It is interesting to note that we may use Equation (C.11) and write
\[ \hat{e}_1(t) \times \frac{d\hat{e}_1(t)}{dt} = \hat{e}_1(t) \times (\omega(t) \times \hat{e}_1(t)) \]
and
\[ \hat{e}_2(t) \times \frac{d\hat{e}_2(t)}{dt} = \hat{e}_2(t) \times (\omega(t) \times \hat{e}_2(t)) \]
and 
\[ \mathbf{\hat{e}}_3(t) \times \frac{d\mathbf{\hat{e}}_3(t)}{dt} = \mathbf{\hat{e}}_3(t) \times (\mathbf{\omega}(t) \times \mathbf{\hat{e}}_3(t)). \]

Then using the vector identity 
\[ \mathbf{F} \times (\mathbf{S} \times \mathbf{T}) = (\mathbf{F} \cdot \mathbf{T})\mathbf{S} - (\mathbf{F} \cdot \mathbf{S})\mathbf{T} \]
these can be expressed as 
\[ \mathbf{\hat{e}}_1(t) \times \frac{d\mathbf{\hat{e}}_1(t)}{dt} = (\mathbf{\hat{e}}_1(t) \cdot \mathbf{\hat{e}}_1(t))\mathbf{\omega}(t) - (\mathbf{\hat{e}}_1(t) \cdot \mathbf{\omega}(t))\mathbf{\hat{e}}_1(t) = \mathbf{\omega}(t) - \omega_1(t)\mathbf{\hat{e}}_1(t) \]
and 
\[ \mathbf{\hat{e}}_2(t) \times \frac{d\mathbf{\hat{e}}_2(t)}{dt} = (\mathbf{\hat{e}}_2(t) \cdot \mathbf{\hat{e}}_2(t))\mathbf{\omega}(t) - (\mathbf{\hat{e}}_2(t) \cdot \mathbf{\omega}(t))\mathbf{\hat{e}}_2(t) = \mathbf{\omega}(t) - \omega_2(t)\mathbf{\hat{e}}_2(t) \]
and 
\[ \mathbf{\hat{e}}_3(t) \times \frac{d\mathbf{\hat{e}}_3(t)}{dt} = (\mathbf{\hat{e}}_3(t) \cdot \mathbf{\hat{e}}_3(t))\mathbf{\omega}(t) - (\mathbf{\hat{e}}_3(t) \cdot \mathbf{\omega}(t))\mathbf{\hat{e}}_3(t) = \mathbf{\omega}(t) - \omega_3(t)\mathbf{\hat{e}}_3(t). \]

Using the fact that 
\[ \mathbf{\hat{e}}_1(t) \cdot \mathbf{\hat{e}}_1(t) = \mathbf{\hat{e}}_2(t) \cdot \mathbf{\hat{e}}_2(t) = \mathbf{\hat{e}}_3(t) \cdot \mathbf{\hat{e}}_3(t) = 1, \]
and adding these give 
\[ \sum_{k=1}^{3} \mathbf{\hat{e}}_k(t) \times \frac{d\mathbf{\hat{e}}_k(t)}{dt} = 3\mathbf{\omega}(t) - (\omega_1(t)\mathbf{\hat{e}}_1(t) + \omega_2(t)\mathbf{\hat{e}}_2(t) + \omega_3(t)\mathbf{\hat{e}}_3(t)) \]
or 
\[ \sum_{k=1}^{3} \mathbf{\hat{e}}_k(t) \times \frac{d\mathbf{\hat{e}}_k(t)}{dt} = 3\mathbf{\omega}(t) - \mathbf{\omega}(t) \]
so that 
\[ \mathbf{\omega}(t) = \frac{1}{2} \sum_{k=1}^{3} \left( \mathbf{\hat{e}}_k(t) \times \frac{d\mathbf{\hat{e}}_k(t)}{dt} \right) \]
or (with the \( t \) dependence suppressed), 
\[ \mathbf{\omega} = \frac{1}{2} \left( \mathbf{\hat{e}}_1 \times \frac{d\mathbf{\hat{e}}_1}{dt} + \mathbf{\hat{e}}_2 \times \frac{d\mathbf{\hat{e}}_2}{dt} + \mathbf{\hat{e}}_3 \times \frac{d\mathbf{\hat{e}}_3}{dt} \right), \quad (C.13b) \]
which is a more symmetric expression for \( \omega(t) \). Finally, we note that going back to Equation (C.5) and using Equation (C.11), we have

\[
\frac{dF(t)}{dt} = \frac{dF_1(t)}{dt} \hat{e}_1(t) + \frac{dF_2(t)}{dt} \hat{e}_2(t) + \frac{dF_3(t)}{dt} \hat{e}_3(t) \\
+ F_1(t) \frac{d\hat{e}_1(t)}{dt} + F_2(t) \frac{d\hat{e}_2(t)}{dt} + F_3(t) \frac{d\hat{e}_3(t)}{dt} \\
+ F_1(t)(\omega(t) \times \hat{e}_1(t)) + F_2(t)(\omega(t) \times \hat{e}_2(t)) + F_3(t)(\omega(t) \times \hat{e}_3(t)) \\
= \frac{dF_1(t)}{dt} \hat{e}_1(t) + \frac{dF_2(t)}{dt} \hat{e}_2(t) + \frac{dF_3(t)}{dt} \hat{e}_3(t) \\
+ \omega(t) \times (F_1(t)\hat{e}_1(t) + F_2(t)\hat{e}_2(t) + F_3(t)\hat{e}_3(t)) \\
\]

or simply

\[
\dot{F}(t) = \dot{F}_1(t)\hat{e}_1(t) + \dot{F}_2(t)\hat{e}_2(t) + \dot{F}_3(t)\hat{e}_3(t) + \omega(t) \times F(t). \tag{C.14}
\]

C.4b A Summary of Results

To summarize the above results, we see (with the \( t \) dependence suppressed) that

\[
\frac{d\hat{e}_i}{dt} = \omega \times \hat{e}_i \tag{C.15a}
\]

for \( i = 1, 2, 3 \), with

\[
\omega = \left( \frac{d\hat{e}_2}{dt} \cdot \hat{e}_3 \right) \hat{e}_1 + \left( \frac{d\hat{e}_3}{dt} \cdot \hat{e}_1 \right) \hat{e}_2 + \left( \frac{d\hat{e}_1}{dt} \cdot \hat{e}_2 \right) \hat{e}_3. \tag{C.15b}
\]

or

\[
\omega = \frac{1}{2} \left( \hat{e}_1 \times \frac{d\hat{e}_1}{dt} + \hat{e}_2 \times \frac{d\hat{e}_2}{dt} + \hat{e}_3 \times \frac{d\hat{e}_3}{dt} \right), \tag{C.15c}
\]

and for

\[
F = F_1\hat{e}_1 + F_2\hat{e}_2 + F_3\hat{e}_3 \tag{C.15d}
\]

we have

\[
\dot{F} = \dot{F}_1\hat{e}_1 + \dot{F}_2\hat{e}_2 + \dot{F}_3\hat{e}_3 + F_1 \frac{d\hat{e}_1}{dt} + F_2 \frac{d\hat{e}_2}{dt} + F_3 \frac{d\hat{e}_3}{dt}. \tag{C.15e}
\]
or
\[ \ddot{\mathbf{F}} = (\dot{F}_1 \mathbf{e}_1 + \dot{F}_2 \mathbf{e}_2 + \dot{F}_3 \mathbf{e}_3) + \mathbf{\omega} \times \mathbf{F}. \]  \hspace{1cm} (C.15f)

We shall shortly see that \( \mathbf{\omega} \) may be interpreted as an angular velocity vector.

\textit{C.4c Second Derivatives}

We may also compute \( \dddot{\mathbf{F}} \) using
\[
\dddot{\mathbf{F}} = \frac{d}{dt}(\ddot{F}_1 \mathbf{e}_1 + \ddot{F}_2 \mathbf{e}_2 + \ddot{F}_3 \mathbf{e}_3) + \frac{d}{dt}(\mathbf{\omega} \times \mathbf{F})
\]
\[
= (\ddot{F}_1 \mathbf{e}_1 + \ddot{F}_2 \mathbf{e}_2 + \ddot{F}_3 \mathbf{e}_3) + \ddot{F}_1 \frac{d\mathbf{e}_1}{dt} + \ddot{F}_2 \frac{d\mathbf{e}_2}{dt} + \ddot{F}_3 \frac{d\mathbf{e}_3}{dt}
\]
\[
+ \frac{d\mathbf{\omega}}{dt} \times \mathbf{F} + \mathbf{\omega} \times \frac{d\mathbf{F}}{dt}
\]
\[
= (\ddot{F}_1 \mathbf{e}_1 + \ddot{F}_2 \mathbf{e}_2 + \ddot{F}_3 \mathbf{e}_3) + \ddot{F}_1 (\mathbf{\omega} \times \mathbf{e}_1) + \ddot{F}_2 (\mathbf{\omega} \times \mathbf{e}_2) + \ddot{F}_3 (\mathbf{\omega} \times \mathbf{e}_3)
\]
\[
+ \frac{d\mathbf{\omega}}{dt} \times \mathbf{F} + \mathbf{\omega} \times ((\ddot{F}_1 \mathbf{e}_1 + \ddot{F}_2 \mathbf{e}_2 + \ddot{F}_3 \mathbf{e}_3) + \mathbf{\omega} \times \mathbf{F})
\]
\[
= (\ddot{F}_1 \mathbf{e}_1 + \ddot{F}_2 \mathbf{e}_2 + \ddot{F}_3 \mathbf{e}_3) + \mathbf{\omega} \times (\ddot{F}_1 \mathbf{e}_1 + \ddot{F}_2 \mathbf{e}_2 + \ddot{F}_3 \mathbf{e}_3) + \mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{F})
\]

or
\[
\dddot{\mathbf{F}} = (\ddot{F}_1 \mathbf{e}_1 + \ddot{F}_2 \mathbf{e}_2 + \ddot{F}_3 \mathbf{e}_3) + 2\mathbf{\omega} \times (\dddot{F}_1 \mathbf{e}_1 + \dddot{F}_2 \mathbf{e}_2 + \dddot{F}_3 \mathbf{e}_3)
\]
\[
+ \frac{d\mathbf{\omega}}{dt} \times \mathbf{F} + \mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{F}). \]  \hspace{1cm} (C.16)

But
\[ \dot{F}_1 \mathbf{e}_1 + \dot{F}_2 \mathbf{e}_2 + \dot{F}_3 \mathbf{e}_3 = \mathbf{\dot{F}} - \mathbf{\omega} \times \mathbf{F} \]

and so
\[
\dddot{\mathbf{F}} = (\dddot{F}_1 \mathbf{e}_1 + \dddot{F}_2 \mathbf{e}_2 + \dddot{F}_3 \mathbf{e}_3) + 2\mathbf{\omega} \times (\mathbf{\dot{F}} - \mathbf{\omega} \times \mathbf{\dot{F}})
\]
\[
+ \frac{d\mathbf{\omega}}{dt} \times \mathbf{F} + \mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{F}).
\]

or
\[ \dddot{\mathbf{F}} = (\dddot{F}_1 \mathbf{e}_1 + \dddot{F}_2 \mathbf{e}_2 + \dddot{F}_3 \mathbf{e}_3) + \mathbf{\alpha} \times \mathbf{F} + 2(\mathbf{\omega} \times \mathbf{\dot{F}}) - \mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{F}) \]

or
\[ \dddot{\mathbf{F}} = (\dddot{F}_1 \mathbf{e}_1 + \dddot{F}_2 \mathbf{e}_2 + \dddot{F}_3 \mathbf{e}_3) + \mathbf{\alpha} \times \mathbf{F} + 2(\mathbf{\omega} \times \mathbf{\dot{F}}) - \mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{F}) \]
where

\[ \alpha \equiv \frac{d\omega}{dt} = \dot{\omega}_1 \hat{e}_1 + \dot{\omega}_2 \hat{e}_2 + \dot{\omega}_3 \hat{e}_3 + \omega_1 \frac{d\hat{e}_1}{dt} + \omega_2 \frac{d\hat{e}_2}{dt} + \omega_3 \frac{d\hat{e}_3}{dt} \]

\[ = \dot{\omega}_1 \hat{e}_1 + \dot{\omega}_2 \hat{e}_2 + \dot{\omega}_3 \hat{e}_3 + \omega_1 (\omega \times \hat{e}_1) + \omega_2 (\omega \times \hat{e}_2) + \omega_3 (\omega \times \hat{e}_3) \]

\[ = \dot{\omega}_1 \hat{e}_1 + \dot{\omega}_2 \hat{e}_2 + \dot{\omega}_3 \hat{e}_3 + \omega \times \omega \]

or simply

\[ \alpha = \dot{\omega}_1 \hat{e}_1 + \dot{\omega}_2 \hat{e}_2 + \dot{\omega}_3 \hat{e}_3. \]

Thus we see that the second derivative of a vector function

\[ F = F_1 \hat{e}_1 + F_2 \hat{e}_2 + F_3 \hat{e}_3 \]

(C.17a)

can be written as

\[ \ddot{F} = (\ddot{F}_1 \hat{e}_1 + \ddot{F}_2 \hat{e}_2 + \ddot{F}_3 \hat{e}_3) + \alpha \times F + 2(\omega \times \dot{F}) - \omega \times (\omega \times F) \]

(C.17b)

with \( \omega = \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3 \) defined by

\[ \omega = \left( \frac{d\hat{e}_2}{dt} \cdot \hat{e}_3 \right) \hat{e}_1 + \left( \frac{d\hat{e}_3}{dt} \cdot \hat{e}_1 \right) \hat{e}_2 + \left( \frac{d\hat{e}_1}{dt} \cdot \hat{e}_2 \right) \hat{e}_3. \]

(C.17c)

or

\[ \omega = \frac{1}{2} \left( \hat{e}_1 \times \frac{d\hat{e}_1}{dt} + \hat{e}_2 \times \frac{d\hat{e}_2}{dt} + \hat{e}_3 \times \frac{d\hat{e}_3}{dt} \right) \]

(C.17d)

and with \( \alpha \) defined as

\[ \alpha = \frac{d\omega}{dt} = \dot{\omega}_1 \hat{e}_1 + \dot{\omega}_2 \hat{e}_2 + \dot{\omega}_3 \hat{e}_3. \]

(C.17e)

We shall shortly see that \( \omega \) may be interpreted as an angular velocity vector and \( \alpha \) may be interpreted as the corresponding angular acceleration vector.

### C.5 Differentiation Within Various Coordinate Systems

We now apply the ideas developed in the previous section to rectangular, cylindrical and spherical coordinates.
**C.5a Differentiation Within Rectangular Coordinates**

If \( \mathbf{F} \) (with the \( t \) dependence suppressed) is represented in rectangular coordinates \((x, y, z)\) as,

\[
\mathbf{F} = F_x \hat{e}_x + F_y \hat{e}_y + F_z \hat{e}_z
\]  
(C.18a)

then each of

\[
\{\hat{e}_1, \hat{e}_2, \hat{e}_3\} = \{\hat{x}, \hat{y}, \hat{z}\}
\]

form a SRT and are constant, so that Equation (C.13a,b) give \( \omega = 0 \) and \( \alpha = 0 \), and then Equations (C.15a-f) give simply

\[
\dot{\mathbf{F}} = \dot{F}_x \hat{e}_x + \dot{F}_y \hat{e}_y + \dot{F}_z \hat{e}_z.
\]  
(C.18b)

while Equations (C.17a-e) give

\[
\ddot{\mathbf{F}} = \ddot{F}_x \hat{e}_x + \ddot{F}_y \hat{e}_y + \ddot{F}_z \hat{e}_z.
\]  
(C.18c)

Note that in the special case when

\[
\mathbf{F} = \mathbf{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z
\]

is the position vector of a particle moving through space, then

\[
\mathbf{v} = \dot{\mathbf{r}} = \dot{x}\hat{e}_x + \dot{y}\hat{e}_y + \dot{z}\hat{e}_z
\]

and

\[
\mathbf{a} = \ddot{\mathbf{r}} = \ddot{x}\hat{e}_x + \ddot{y}\hat{e}_y + \ddot{z}\hat{e}_z
\]

give the velocity and acceleration, respectively, of this particle.

**C.5b Differentiation Within Cylindrical Coordinates**

If \( \mathbf{F} \) (with the \( t \) dependence suppressed) is represented in cylindrical coordinates \((\rho, \theta, z)\) as,

\[
\mathbf{F} = F_\rho \hat{e}_\rho + F_\theta \hat{e}_\theta + F_z \hat{e}_z
\]  
(C.19a)

then the unit vectors

\[
\{\hat{e}_1, \hat{e}_2, \hat{e}_3\} = \{\hat{\rho}, \hat{\theta}, \hat{z}\}
\]
form a SRT and are related to the unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ by

$$\mathbf{e}_\rho = \cos(\theta)\mathbf{e}_x + \sin(\theta)\mathbf{e}_y$$

and

$$\mathbf{e}_\theta = -\sin(\theta)\mathbf{e}_x + \cos(\theta)\mathbf{e}_y$$

which in matrix form reads

$$\begin{bmatrix} \hat{\mathbf{e}}_\rho \\ \hat{\mathbf{e}}_\theta \\ \hat{\mathbf{e}}_z \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z \end{bmatrix}$$  \hspace{2cm} (C.20)

where $\theta = \theta(t)$ is, in general, a function of $t$. Thus we have

$$\frac{d\mathbf{e}_\rho}{dt} = -\dot{\theta}\sin(\theta)\mathbf{e}_x + \dot{\theta}\cos(\theta)\mathbf{e}_y = \dot{\theta}\mathbf{e}_\theta$$

and

$$\frac{d\mathbf{e}_\theta}{dt} = -\dot{\theta}\cos(\theta)\mathbf{e}_x - \dot{\theta}\sin(\theta)\mathbf{e}_y = -\dot{\theta}\mathbf{e}_\rho$$

and

$$\frac{d\mathbf{e}_z}{dt} = 0.$$

Then from Equation (C.13a), we have

$$\begin{align*}
\mathbf{\omega} &= \left(\frac{d\mathbf{e}_\theta}{dt} \cdot \mathbf{e}_z\right) \mathbf{e}_\rho + \left(\frac{d\mathbf{e}_z}{dt} \cdot \mathbf{e}_\rho\right) \mathbf{e}_\theta + \left(\frac{d\mathbf{e}_\rho}{dt} \cdot \mathbf{e}_\theta\right) \mathbf{e}_z \\
&= (-\dot{\theta}\mathbf{e}_\rho \cdot \mathbf{e}_z) \mathbf{e}_\rho + (0 \cdot \mathbf{e}_\rho) \mathbf{e}_\theta + \left(\dot{\theta}\mathbf{e}_\theta \cdot \mathbf{e}_\theta\right) \mathbf{e}_z \\
&= (0) \mathbf{e}_\rho + (0) \mathbf{e}_\theta + (\dot{\theta}) \mathbf{e}_z
\end{align*}$$

or simply

$$\mathbf{\omega} = \dot{\theta}\mathbf{e}_z.$$ \hspace{2cm} (C.21a)

Then Equations (C.15a-f) give

$$\begin{align*}
\mathbf{\dot{F}} &= \dot{F}_\rho \mathbf{e}_\rho + \dot{F}_\theta \mathbf{e}_\theta + \dot{F}_z \mathbf{e}_z + F_\rho \frac{d\mathbf{e}_\rho}{dt} + F_\theta \frac{d\mathbf{e}_\theta}{dt} + F_z \frac{d\mathbf{e}_z}{dt} \\
&= \dot{F}_\rho \mathbf{e}_\rho + \dot{F}_\theta \mathbf{e}_\theta + \dot{F}_z \mathbf{e}_z + F_\rho(\dot{\theta}\mathbf{e}_\rho) + F_\theta(-\dot{\theta}\mathbf{e}_\rho) + F_z(0)
\end{align*}$$

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or simply
\[ \ddot{\mathbf{F}} = (\dot{\mathbf{F}}_\rho - \dot{\theta} F_\theta) \mathbf{e}_\rho + (\dot{\mathbf{F}}_\theta + \dot{\theta} F_\rho) \mathbf{e}_\theta + \ddot{\mathbf{F}}_z \mathbf{e}_z. \]  
(C.21b)

We also have
\[ \ddot{\mathbf{F}} = (\ddot{\mathbf{F}}_\rho \mathbf{e}_\rho + \ddot{\mathbf{F}}_\theta \mathbf{e}_\theta + \ddot{\mathbf{F}}_z \mathbf{e}_z) + \alpha \times \mathbf{F} + 2(\omega \times \dot{\mathbf{F}}) - \omega \times (\omega \times \mathbf{F}) \]
with
\[ \alpha = \ddot{\theta} \mathbf{e}_z. \]

This reduces to
\[
\ddot{\mathbf{F}} = (\ddot{\mathbf{F}}_\rho \mathbf{e}_\rho + \ddot{\mathbf{F}}_\theta \mathbf{e}_\theta + \ddot{\mathbf{F}}_z \mathbf{e}_z) + \ddot{\theta} \mathbf{e}_z \times ((\dot{\mathbf{e}}_z) \times (F_\rho \mathbf{e}_\rho + F_\theta \mathbf{e}_\theta + F_z \mathbf{e}_z)) \\
+ 2(\ddot{\mathbf{e}}_z \times ((\dot{\mathbf{F}}_\rho - \dot{\theta} F_\theta) \mathbf{e}_\rho + (\dot{\theta} F_\rho) \mathbf{e}_\theta + \ddot{\mathbf{F}}_z \mathbf{e}_z)) \\
- \ddot{\theta} \mathbf{e}_z \times ((\dot{\mathbf{e}}_z) \times (F_\rho \mathbf{e}_\rho + F_\theta \mathbf{e}_\theta + F_z \mathbf{e}_z)) \\
= (\ddot{\mathbf{F}}_\rho \mathbf{e}_\rho + \ddot{\mathbf{F}}_\theta \mathbf{e}_\theta + \ddot{\mathbf{F}}_z \mathbf{e}_z) + \ddot{\mathbf{F}}_\rho \mathbf{e}_\rho - \ddot{\theta} F_\rho \mathbf{e}_\rho \\
+ 2(\ddot{\mathbf{F}}_\rho \dot{\theta} - F_\rho \dot{\theta}^2) \mathbf{e}_\theta - 2(\dot{\mathbf{F}}_\theta \dot{\theta} + F_\rho \dot{\theta}^2) \mathbf{e}_\rho \\
- \ddot{\theta} \mathbf{e}_z \times ((\dot{\mathbf{F}}_\rho \mathbf{e}_\rho - \dot{\theta} F_\rho \mathbf{e}_\rho)) \\
\]

or simply
\[ \ddot{\mathbf{F}} = (\ddot{\mathbf{F}}_\rho - \ddot{\theta} F_\rho - 2\dot{\theta} \ddot{F}_\theta - \dot{\theta}^2 F_\rho) \mathbf{e}_\rho \\
+ (\ddot{\mathbf{F}}_\theta + \ddot{\theta} F_\rho + 2\dot{\theta} \dot{\theta} F_\rho - \dot{\theta}^2 F_\rho) \mathbf{e}_\theta + \ddot{\mathbf{F}}_z \mathbf{e}_z. \]  
(C.21c)

Note that in the special case when
\[ \mathbf{F} = \mathbf{r} = \rho \mathbf{e}_\rho + 0 \mathbf{e}_\theta + z \mathbf{e}_z \]
is the position vector of a particle moving through space, then
\[ \mathbf{v} = \dot{\mathbf{r}} = \dot{\rho} \mathbf{e}_\rho + \rho \dot{\theta} \mathbf{e}_\theta + \dot{z} \mathbf{e}_z \]
and
\[ \mathbf{a} = \ddot{\mathbf{r}} = (\ddot{\rho} - \rho \ddot{\theta}^2) \mathbf{e}_\rho + (\rho \ddot{\theta} + 2\dot{\rho} \dot{\theta}) \mathbf{e}_\theta + \ddot{z} \mathbf{e}_z \]
give the velocity and acceleration, respectively, of this particle.
C.5c Differentiation Within Spherical Coordinates

If \( \mathbf{F} \) (with the \( t \) dependence suppressed) is represented in spherical coordinates \((r, \varphi, \theta)\) as,

\[ \mathbf{F} = F_r \hat{e}_r + F_\varphi \hat{e}_\varphi + F_\theta \hat{e}_\theta \]  

(C.22)

then the unit vectors \( \{\hat{e}_1, \hat{e}_2, \hat{e}_3\} = \{\hat{e}_r, \hat{e}_\varphi, \hat{e}_\theta\} \) form a SRT and are related to the unit vectors \( \hat{e}_\rho, \hat{e}_\theta \) and \( \hat{e}_z \) by

\[ \hat{e}_r = \sin(\varphi)\hat{e}_\rho + \cos(\varphi)\hat{e}_z \]

and

\[ \hat{e}_\varphi = \cos(\varphi)\hat{e}_\rho - \sin(\varphi)\hat{e}_z \]

and

\[ \hat{e}_\theta = \hat{e}_\theta \]

which in matrix form reads

\[
\begin{bmatrix}
\hat{e}_r \\
\hat{e}_\varphi \\
\hat{e}_\theta
\end{bmatrix}
= \begin{bmatrix}
\sin(\varphi) & 0 & \cos(\varphi) \\
\cos(\varphi) & 0 & -\sin(\varphi) \\
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
\hat{e}_\rho \\
\hat{e}_\theta \\
\hat{e}_z
\end{bmatrix},
\]  

(C.23)

where \( \varphi = \varphi(t) \) is, in general, a function of \( t \). Putting in Equation (C.20), we then see that the unit vectors \( \hat{e}_r, \hat{e}_\varphi, \hat{e}_\theta \) are related to the unit vectors \( \hat{e}_x, \hat{e}_y \) and \( \hat{e}_z \) by

\[
\begin{bmatrix}
\hat{e}_r \\
\hat{e}_\varphi \\
\hat{e}_\theta
\end{bmatrix}
= \begin{bmatrix}
\sin(\varphi) & 0 & \cos(\varphi) \\
\cos(\varphi) & 0 & -\sin(\varphi) \\
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
\cos(\theta) & \sin(\theta) & 0 \\
-\sin(\theta) & \cos(\theta) & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\hat{e}_x \\
\hat{e}_y \\
\hat{e}_z
\end{bmatrix},
\]

or

\[
\begin{bmatrix}
\hat{e}_r \\
\hat{e}_\varphi \\
\hat{e}_\theta
\end{bmatrix}
= \begin{bmatrix}
\sin(\varphi) & 0 & \cos(\varphi) \\
\cos(\varphi) & 0 & -\sin(\varphi) \\
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
\hat{e}_x \\
\hat{e}_y \\
\hat{e}_z
\end{bmatrix},
\]  

(C.24)

where \( \theta = \theta(t) \) and \( \varphi = \varphi(t) \) are, in general, functions of \( t \). Then

\[ \hat{e}_r = \sin(\varphi)\cos(\theta)\hat{e}_x + \sin(\varphi)\sin(\theta)\hat{e}_y + \cos(\varphi)\hat{e}_z \]
and
\[ \hat{\mathbf{e}}_\varphi = \cos(\varphi) \cos(\theta) \hat{\mathbf{e}}_x + \cos(\varphi) \sin(\theta) \hat{\mathbf{e}}_y - \sin(\varphi) \hat{\mathbf{e}}_z \]
and
\[ \hat{\mathbf{e}}_\theta = -\sin(\theta) \hat{\mathbf{e}}_x + \cos(\theta) \hat{\mathbf{e}}_y \]
so that
\[
\frac{d\hat{\mathbf{e}}_r}{dt} = (\dot{\varphi} \cos(\varphi) \cos(\theta) - \dot{\theta} \sin(\varphi) \sin(\theta)) \hat{\mathbf{e}}_x \\
+ (\dot{\varphi} \sin(\varphi) \sin(\theta) + \dot{\theta} \sin(\varphi) \cos(\theta)) \hat{\mathbf{e}}_y - \dot{\varphi} \sin(\varphi) \hat{\mathbf{e}}_z \\
= \dot{\varphi} (\cos(\varphi) \cos(\theta) \hat{\mathbf{e}}_x + \sin(\varphi) \sin(\theta) \hat{\mathbf{e}}_y - \sin(\varphi) \hat{\mathbf{e}}_z) \\
+ \dot{\theta} \sin(\varphi) (-\sin(\theta) \hat{\mathbf{e}}_x + \cos(\theta) \hat{\mathbf{e}}_y)
\]
or simply
\[
\frac{d\hat{\mathbf{e}}_r}{dt} = \dot{\varphi} \hat{\mathbf{e}}_\varphi + \dot{\theta} \sin(\varphi) \hat{\mathbf{e}}_\theta. \tag{C.25a}
\]
We also have
\[
\frac{d\hat{\mathbf{e}}_\varphi}{dt} = (-\dot{\varphi} \sin(\varphi) \cos(\theta) - \dot{\theta} \cos(\varphi) \sin(\theta)) \hat{\mathbf{e}}_x \\
+ (-\dot{\varphi} \sin(\varphi) \sin(\theta) + \dot{\theta} \cos(\varphi) \cos(\theta)) \hat{\mathbf{e}}_y - \dot{\varphi} \cos(\varphi) \hat{\mathbf{e}}_z \\
= -\dot{\varphi} (\sin(\varphi) \cos(\theta) \hat{\mathbf{e}}_x + \sin(\varphi) \sin(\theta) \hat{\mathbf{e}}_y + \cos(\varphi) \hat{\mathbf{e}}_z) \\
+ \dot{\theta} \cos(\varphi) (-\sin(\theta) \hat{\mathbf{e}}_x + \cos(\theta) \hat{\mathbf{e}}_y)
\]
or simply
\[
\frac{d\hat{\mathbf{e}}_\varphi}{dt} = -\dot{\varphi} \hat{\mathbf{e}}_r + \dot{\theta} \cos(\varphi) \hat{\mathbf{e}}_\theta. \tag{C.25b}
\]
Finally, we also have
\[
\frac{d\hat{\mathbf{e}}_\theta}{dt} = -\dot{\theta} \cos(\theta) \hat{\mathbf{e}}_x - \dot{\theta} \sin(\theta) \hat{\mathbf{e}}_y = -\dot{\theta} \hat{\mathbf{e}}_\rho.
\]
Using the inverse of Equation (C.23),
\[
\begin{bmatrix}
\hat{\mathbf{e}}_\rho \\
\hat{\mathbf{e}}_\theta \\
\hat{\mathbf{e}}_z
\end{bmatrix} =
\begin{bmatrix}
\sin(\varphi) & 0 & \cos(\varphi) \\
\cos(\varphi) & 0 & -\sin(\varphi) \\
0 & 1 & 0
\end{bmatrix}^{-1}
\begin{bmatrix}
\hat{\mathbf{e}}_r \\
\hat{\mathbf{e}}_\varphi \\
\hat{\mathbf{e}}_\theta
\end{bmatrix}
\]

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or
\[
\begin{bmatrix}
\hat{e}_ρ \\
\hat{e}_θ \\
\hat{e}_ϕ
\end{bmatrix} =
\begin{bmatrix}
\sin(\varphi) & \cos(\varphi) & 0 \\
0 & 0 & 1 \\
\cos(\varphi) & -\sin(\varphi) & 0
\end{bmatrix}
\begin{bmatrix}
\hat{e}_r \\
\hat{e}_ϕ \\
\hat{e}_θ
\end{bmatrix}
\]
we see that
\[
\hat{e}_ρ = \sin(\varphi)\hat{e}_r + \cos(\varphi)\hat{e}_ϕ
\]
and putting this into the above equation for \(d\hat{e}_θ/dt\), we see that
\[
\frac{d\hat{e}_θ}{dt} = -\dot{\varphi}\sin(\varphi)\hat{e}_r - \dot{\vartheta}\cos(\varphi)\hat{e}_ϕ. \tag{C.25c}
\]
Using Equation (C.13a), we now find that
\[
\omega = \left(\frac{d\hat{e}_ϕ}{dt} \cdot \hat{e}_θ\right) \hat{e}_r + \left(\frac{d\hat{e}_θ}{dt} \cdot \hat{e}_r\right) \hat{e}_ϕ + \left(\frac{d\hat{e}_r}{dt} \cdot \hat{e}_ϕ\right) \hat{e}_θ
\]
which reduces to
\[
\omega = \left((-\dot{\varphi}\hat{e}_r + \dot{\vartheta}\cos(\varphi)\hat{e}_θ) \cdot \hat{e}_r\right) \hat{e}_r \\
+ \left((-\dot{\vartheta}\sin(\varphi)\hat{e}_r - \dot{\varphi}\cos(\varphi)\hat{e}_ϕ) \cdot \hat{e}_r\right) \hat{e}_ϕ \\
+ \left((\dot{\varphi}\hat{e}_ϕ + \dot{\vartheta}\sin(\varphi)\hat{e}_θ) \cdot \hat{e}_ϕ\right) \hat{e}_θ
\]
or simply
\[
\omega = \dot{\vartheta}\cos(\varphi)\hat{e}_ϕ - \dot{\varphi}\sin(\varphi)\hat{e}_ϕ + \dot{\varphi}\hat{e}_θ. \tag{C.26}
\]
Then Equations (C.15a-f) give
\[
\dot{F} = \dot{F}_r\hat{e}_r + \dot{F}_ϕ\hat{e}_ϕ + \dot{F}_θ\hat{e}_θ + F_r\frac{d\hat{e}_r}{dt} + F_ϕ\frac{d\hat{e}_ϕ}{dt} + F_θ\frac{d\hat{e}_θ}{dt}
\]
\[
= \dot{F}_r\hat{e}_r + \dot{F}_ϕ\hat{e}_ϕ + \dot{F}_θ\hat{e}_θ + F_r(\dot{\varphi}\hat{e}_ϕ + \dot{\vartheta}\sin(\varphi)\hat{e}_θ) \\
+ F_ϕ(-\dot{\varphi}\hat{e}_r + \dot{\vartheta}\cos(\varphi)\hat{e}_θ) + F_θ(-\dot{\varphi}\sin(\varphi)\hat{e}_r - \dot{\vartheta}\cos(\varphi)\hat{e}_ϕ)
\]
or simply
\[
\dot{F} = (\dot{F}_r - \dot{\varphi}F_ϕ - \dot{\vartheta}F_θ\sin(\varphi))\hat{e}_r + (\dot{\varphi}F_r + \dot{F}_ϕ - \dot{\vartheta}F_θ\cos(\varphi))\hat{e}_ϕ \\
+ (\dot{\varphi}F_ϕ \sin(\varphi) + \dot{\vartheta}F_ϕ \cos(\varphi) + \dot{F}_θ)\hat{e}_θ.
\]
Thus for
\[ F = F_r \hat{e}_r + F_\varphi \hat{e}_\varphi + F_\theta \hat{e}_\theta \]  
we have
\[
\begin{align*}
\dot{F} &= (\dot{F}_r - \varphi \dot{F}_\varphi - \dot{\theta} F_\theta \sin(\varphi))\hat{e}_r + (\varphi \dot{F}_r + \dot{F}_\varphi - \dot{\theta} F_\theta \cos(\varphi))\hat{e}_\varphi \\
&\quad + (\dot{\theta} F_r \sin(\varphi) + \dot{\theta} F_\varphi \cos(\varphi) + \dot{F}_\theta)\hat{e}_\theta. 
\end{align*} 
\]  
(C.27b)

We leave it as an exercise to compute \( \ddot{F} \), and you should find (after much algebra) that
\[
\begin{align*}
\ddot{F} &= \{ \ddot{F}_r - \dot{\varphi}^2 F_r - \ddot{\varphi} F_\varphi - 2 \dot{\varphi} \ddot{F}_\varphi - (\ddot{\theta} F_\theta + 2 \dot{\theta} \dot{F}_\theta) \sin(\varphi) \\
&\quad - \dot{\theta}^2 F_r \sin^2(\varphi) - \dot{\theta}^2 F_\varphi \sin(\varphi) \cos(\varphi) \} \hat{e}_r \\
&\quad + \{ \ddot{\varphi} F_r + \ddot{F}_r - \dot{\varphi}^2 F_\varphi - (\ddot{\theta} F_\theta + 2 \dot{\theta} \dot{F}_\theta) \cos(\varphi) \\
&\quad - \dot{\theta}^2 F_r \sin(\varphi) \cos(\varphi) - \dot{\theta}^2 F_\varphi \cos^2(\varphi) \} \hat{e}_\varphi \\
&\quad + \{ \ddot{F}_\theta - \dot{\theta}^2 F_\theta + (\ddot{\theta} F_r + 2 \dot{\theta} \dot{F}_r - 2 \dot{\varphi} \dot{\theta} F_\varphi) \sin(\varphi) \\
&\quad + (2 \ddot{\varphi} \dot{F}_r + \dot{\theta} F_\varphi + 2 \dot{\theta} \dot{F}_\varphi) \cos(\varphi) \} \hat{e}_\theta. 
\end{align*} 
\]  
(C.27c)

Note that in the special case when
\[ F = \mathbf{r} = r \hat{e}_r + 0 \hat{e}_\varphi + 0 \hat{e}_\theta \]
is the position vector of a particle moving through space, then
\[ \mathbf{v} = \dot{\mathbf{r}} = \dot{r} \hat{e}_r + r \dot{\varphi} \hat{e}_\varphi + r \dot{\theta} \sin(\varphi) \hat{e}_\theta. \]

and
\[ \mathbf{a} = \ddot{\mathbf{r}} = (\ddot{r} - r \dot{\varphi}^2 - r \dot{\theta}^2 \sin^2(\varphi)) \hat{e}_r \\
\quad + (r \ddot{\varphi} + 2 \dot{r} \dot{\varphi} - r \dot{\theta}^2 \sin(\varphi) \cos(\varphi)) \hat{e}_\varphi \\
\quad + ((r \ddot{\theta} + 2 \dot{r} \dot{\theta}) \sin(\varphi) + 2 r \dot{\varphi} \dot{\theta} \cos(\varphi)) \hat{e}_\theta. \]
give the velocity and acceleration, respectively, of this particle.
C.6 Construction of a SRT from any Vector Function $G = G(t)$

We end this appendix by noting that one may construct a SRT by starting with any vector function $G = G(t)$ for which $G(t) \neq 0$ for all $t$. We begin by simply setting

$$\hat{e}_1(t) \equiv \frac{G(t)}{|G(t)|}. \quad (C.28a)$$

Then we compute

$$\frac{d\hat{e}_1(t)}{dt} = \frac{1}{|G(t)|^2} \left\{ |G(t)| \frac{dG(t)}{dt} - \frac{d|G(t)|}{dt} G(t) \right\}$$

which must be perpendicular to $\hat{e}_2(t)$, and so, as long as

$$\frac{d\hat{e}_1(t)}{dt} \neq 0$$

for all $t$, we may set

$$\hat{e}_2(t) \equiv \frac{d\hat{e}_1(t)/dt}{|d\hat{e}_1(t)/dt|}. \quad (C.28b)$$

Finally, we set

$$\hat{e}_3(t) \equiv \hat{e}_1(t) \times \hat{e}_2(t) \quad (C.28c)$$

and the set

$$S\{t\} = \{\hat{e}_1(t), \hat{e}_2(t), \hat{e}_3(t)\}$$

will form a SRT that changes with the variable $t$. These ideas will be applied in the study of curvilinear motion in the next appendix, but let us first look at an example.

**Example**

Suppose that

$$G(t) = (1 + t)^{1/2}\hat{e}_x + \hat{e}_y + (1 - t)^{1/2}\hat{e}_z.$$ 

for $-1 < t < +1$. Then

$$|G(t)| = \sqrt{(1 + t) + 1 + (1 - t)} = \sqrt{3}$$

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and so
\[ \hat{e}_1(t) = \sqrt{\frac{1+t}{3}} \hat{e}_x + \sqrt{\frac{1}{3}} \hat{e}_y + \sqrt{\frac{1-t}{3}} \hat{e}_z. \]

Then
\[ \frac{d\hat{e}_1(t)}{dt} = \frac{1}{2\sqrt{3}} (1+t)^{-1/2} \hat{e}_x + 0 \hat{e}_y - \frac{1}{2\sqrt{3}} (1-t)^{-1/2} \hat{e}_z \]
and
\[ \left| \frac{d\hat{e}_1(t)}{dt} \right| = \sqrt{\frac{1}{12(1+t)} + \frac{1}{12(1-t)}} = \frac{1}{\sqrt{6(1-t^2)}} \]
so that Equation (C.28b) reduces to
\[ \hat{e}_2(t) \equiv \sqrt{\frac{1-t}{2}} \hat{e}_x + 0 \hat{e}_y - \sqrt{\frac{1+t}{2}} \hat{e}_z. \]

Finally, Equation (C.28c) leads to
\[ \hat{e}_3(t) \equiv \hat{e}_1(t) \times \hat{e}_2(t) = \frac{1}{\sqrt{6}} \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \sqrt{1+t} & 1 & \sqrt{1-t} \\ \sqrt{1-t} & 0 & -\sqrt{1+t} \end{vmatrix}, \]
which reduces to
\[ \hat{e}_3(t) \equiv -\sqrt{\frac{1+t}{6}} \hat{e}_x + \sqrt{\frac{1}{6}} \hat{e}_y - \sqrt{\frac{1-t}{6}} \hat{e}_z. \]

Thus we have the SRT
\[ S\{t\} = \{\hat{e}_1(t), \hat{e}_2(t), \hat{e}_3(t)\} \]
that changes with the variable \( t \).