Stability of the Null Solution

Stability in the sense of Lyapunov
\[ x(t) \to 0 \text{ as } t \to t_0 \]
\[ \Rightarrow \| x(t_0) \| < \delta \]
\[ \Rightarrow \| x(t) \| < \varepsilon, \quad t > t_0 \]

Asymptotic Stability
\[ x(t) \to 0 \text{ as } t \to \infty \]
\[ \delta \varepsilon x_1 x_2 \]
\[ x_0 \text{ is said to be attractive!} \]

Linear Autonomous (Time-Independent) Systems

\[ \dot{x} = f(x), \quad f(x) = Ax \]
\[ x \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n \times n} \]
\[ f : \mathbb{R}^n \to \mathbb{R}^n \]

Local Stability implies Global Stability
- Global Asymptotic Stability
  - if and only if the real parts of all eigenvalues are nonpositive, and zero eigenvalue is not repeated
- Lyapunov Stability, not Global Asymptotic Stability
  - if and only if the real parts of all eigenvalues are negative
- Unstable
  - if and only if there is one eigenvalue whose real part is positive

Lyapunov’s theorem

Nonlinear, autonomous systems
Near equilibrium points

If the linearized system exhibits significant behavior, then the stability characteristics of the nonlinear system near the equilibrium point is the same as that of the linear system.
Example 1

Equation of motion
\[ \ddot{q} + \frac{c}{ml^2} \dot{q} + \frac{g}{l} \sin q = 0 \]

State space representation
\[ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{g}{l} \sin x_1 - \frac{c}{ml^2} x_2 \\ x_2 \end{bmatrix} \]

Equilibrium points
\[ x_{e,1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad x_{e,2} = \begin{bmatrix} \pi \\ 0 \end{bmatrix} \]

Change of variables
\[ \tilde{x}_1 = (x - x_{e,1}), \quad \tilde{x}_2 = (x - x_{e,2}) \]

Example 1 (continued)

Equilibrium point number 1
\[ x_{e,1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \tilde{x} = \begin{bmatrix} \tilde{\tilde{x}}_1 \\ \tilde{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} \frac{g}{l} \sin \tilde{x}_1 - \frac{c}{ml^2} \tilde{x}_2 \\ \tilde{x}_2 \end{bmatrix} \]

Linearization
\[ f(\tilde{x}) = f(0) + \begin{bmatrix} \frac{df(\tilde{x})}{d\tilde{x}} \end{bmatrix} \tilde{x} + O(\tilde{x}^2) \]
\[ - A \tilde{x} \]
\[ A = \begin{bmatrix} 0 & \frac{c}{ml^2} \\ - \frac{g}{l} & - \frac{c}{ml^2} \end{bmatrix} \]
\[ \lambda_1,2 = - \frac{c}{2ml^2} \pm \frac{1}{2} \sqrt{\left( \frac{c}{ml^2} \right)^2 + \frac{4g}{l}} \]

The system is locally asymptotically stable.

Example 1 (continued)

Equilibrium point number 2
\[ x_{e,2} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}, \quad \tilde{x} = \begin{bmatrix} \tilde{\tilde{x}}_1 \\ \tilde{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} \frac{g}{l} \sin \tilde{x}_1 - \frac{c}{ml^2} \tilde{x}_2 \\ \tilde{x}_2 \end{bmatrix} \]

Linearization
\[ f(\tilde{x}) = f(0) + \begin{bmatrix} \frac{df(\tilde{x})}{d\tilde{x}} \end{bmatrix} \tilde{x} + O(\tilde{x}^2) \]
\[ - A \tilde{x} \]
\[ A = \begin{bmatrix} 0 & \frac{c}{ml^2} \\ - \frac{g}{l} & - \frac{c}{ml^2} \end{bmatrix} \]
\[ \lambda_1,2 = - \frac{c}{2ml^2} \pm \frac{1}{2} \sqrt{\left( \frac{c}{ml^2} \right)^2 + \frac{4g}{l}} \]

The system is unstable.

Example 1 (continued)

Equilibrium point number 1
\[ x_{e,1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
\[ \tilde{x} = \begin{bmatrix} \tilde{\tilde{x}}_1 \\ \tilde{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} \frac{g}{l} \sin \tilde{x}_1 - \frac{c}{ml^2} \tilde{x}_2 \\ \tilde{x}_2 \end{bmatrix} \]

Equilibrium point number 2
\[ x_{e,2} = \begin{bmatrix} \pi \\ 0 \end{bmatrix} \]
\[ \tilde{x} = \begin{bmatrix} \tilde{\tilde{x}}_1 \\ \tilde{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} \frac{g}{l} \sin \tilde{x}_1 - \frac{c}{ml^2} \tilde{x}_2 \\ \tilde{x}_2 \end{bmatrix} \]
Example 1 \((c=0)\)

Equilibrium point number 1

\[ x_{e1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

\[ \dot{x} = \begin{bmatrix} x_2 \\ -\frac{c}{l} \sin \theta \end{bmatrix} \]

Linearization

\[ f(t) = f(0) + \frac{df(t)}{dt}(0) \cdot (t) + O(t^2) \]

\[ -1 \dot{\theta} \]

\[ A = \begin{bmatrix} 0 & 1 \\ \frac{c}{l} & 0 \end{bmatrix} \]

\[ \lambda_1 = \frac{c}{l} \]

Real parts of both eigenvalues are nonnegative

No conclusive results

Example 2

One-dimensional spring-mass-dashpot with a nonlinear spring

\[ m \ddot{x} + b \dot{x} + kx^3 = 0 \]

Example 3

\[ f = k_1 \theta + k_2 \dot{\theta} \]

\[ M = \frac{1}{2} m \left( \frac{1}{2} \right) \sin(q_2) + \frac{1}{2} m \left( \frac{1}{2} \right) \cos(q_2) \sin(q_2) + K q_1 \left( \frac{1}{2} q_2 - \frac{1}{2} q_2^3 \right) = 0 \]

\[ \frac{1}{6} m b \left( -3 \cos(q_2) \sin(q_2) - 2 b \sin(q_2) + 3 \sin(q_2) \right) = 0 \]

Example 4

A rigid body is undergoing a steady rotation about the intermediate principal axis at its center of mass. That is,

\[ \omega_1 = \omega_3 = 0, \omega_2 = \omega_0 \]

Verify that \( \omega_1 = \omega_3 = 0, \omega_2 = \omega_0 \) is a steady state motion, with \( M' = M, \theta = 0 \)

Write equations in the body-fixed frame, \( B \)

\[ I_{1b} \omega_0 - (I_2 - I_3) \omega_0 \omega_0 = M' \]

\[ I_{3b} \omega_0 - (I_1 - I_3) \omega_0 \omega_0 = M' \]

\[ I_{1b} \omega_1 - (I_1 + I_3) \omega_1 \omega_2 = M' \]

\[ \lambda = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]

\[ \lambda_1 = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{I_1} \left( I_2 - I_3 \right) \end{bmatrix} \]

\[ \lambda_2 = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{I_2} \left( I_1 - I_3 \right) \end{bmatrix} \]

\[ \lambda_3 = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{I_3} \left( I_1 - I_2 \right) \end{bmatrix} \]

\[ A = \begin{bmatrix} 0 & 0 & \frac{1}{I_1} \left( I_2 - I_3 \right) \\ 0 & 0 & \frac{1}{I_2} \left( I_1 - I_3 \right) \\ 0 & 0 & \frac{1}{I_3} \left( I_1 - I_2 \right) \end{bmatrix} \]
Example 4

\[ \omega_1 = \omega_2 = 0, \omega_3 = \omega_0 \]

\[ \lambda = 0 \quad \lambda_{ij} = \omega_j \omega_i + \frac{(I_j - I_i)}{I_{ii}} \]

Significant dynamics
Unstable!

What if \( \omega_1 = \omega_2 = 0, \omega_3 = \omega_0 \) ?

Or \( \omega_2 = \omega_3 = 0, \omega_1 = \omega_0 \) ?

Insignificant dynamics
Cannot establish conclusive results

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Summary for Nonlinear Autonomous Systems

Write equations of motion in state space notation
\[ x = f(x) \]
\[ x \in \mathbb{R}^n, \quad f : \mathbb{R}^n \to \mathbb{R}^n \]

Identify equilibrium point(s), \( x_e \)
Solve \( f(x) = 0 \)

Linearize equations of motion to get the coefficient matrix \( A \)
\[ A = \left( \frac{\partial f}{\partial x} \right)_{x=x_e} \]

Compute eigenvalues of \( A \). Does the linearized system have significant dynamics?

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Lyapunov’s Direct Method

*Avoids linearization (hence direct)
Lyapunov’s Direct Method

$V(x): \mathbb{R}^n \rightarrow \mathbb{R}^n$

$V(0) = 0$

$V(x) > 0, \text{ for } x \neq 0$

- $V(x)$ is a continuous function with continuous first partial derivatives
- $V(x)$ is positive definite

Such a function $V$ is called a Lyapunov Function Candidate

$V$ acts like a norm

What if you can show that $V$ never increases?

Theorem

1. The (above) system is stable if there exists a Lyapunov function candidate such that the time derivative of $V$ is negative semi-definite along all solution trajectories of the system.

$\dot{V}(x) \leq \frac{\partial V}{\partial x} \cdot f(x) \leq 0$

Theorem

2. The (above) system is asymptotically stable if there exists a Lyapunov function candidate such that the time derivative of $V$ is negative definite along all solution trajectories of the system.

$\dot{V}(x) = \frac{\partial V}{\partial x} \cdot f(x) < 0$

Example 1

Equation of motion

$\ddot{q} + \frac{c}{m} \dot{q} + \frac{g}{l} \sin q = 0$

State space representation

$x = [q \ \dot{q}]$

$\dot{x} = \begin{bmatrix} \frac{g}{l} \sin q_1 - \frac{c}{m} \dot{q} \\frac{x_2}{m} \end{bmatrix}$

Equilibrium point

$x_0 = [0 \ 0]$

What is a candidate Lyapunov function?
Example 3

One-dimensional spring-mass-dashpot with a nonlinear spring

\[ m\ddot{x} + b\dot{x} + kx^3 = 0 \]

What is a candidate Lyapunov function?

\[ \frac{1}{2} m(x_2)^2 + \frac{1}{4} k(x_3)^4 \]

Example 4

A rigid body is undergoing a steady rotation about the intermediate principal axis at its center of mass. That is,

\[ \omega_1 = \omega_2 = 0, \quad \omega_3 = \omega_0 \]

Verify that \( \omega_1 = \omega_2 = 0, \omega_3 = \omega_0 \)

is a steady state motion, with \( M \dot{x} = M \dot{y} = M \dot{z} = 0 \)

Example: Precessing top

Nonlinear Time-Varying Systems

Write equations of motion in state space notation

\[ x = f(x,t) \]

\[ x \in \mathbb{R}^n, \quad f : \mathbb{R}^n \to \mathbb{R}^n \]

Identify stationary solutions, \( x(t) \)

Solve \( f(x,t) = 0 \) to get

\( x_s(t) = \phi(t) \)

What happens if the system is perturbed from the stationary motion?

\[ x(t) = \phi(t) + y(t) \]

\[ \dot{\phi}(t) + y(t) = f(\phi(t)) + y(t) \]

\[ \dot{\phi}(t) = f(\phi(t)) \]

stationary solution

perturbation

Stability with respect to perturbations about stationary motions

Compare equations for the perturbed system with the stationary solution:

\[ \dot{\phi}(t) + y(t) = f(\phi(t)) + y(t) x(t) \]

\[ \dot{\phi}(t) = f(\phi(t)) x(t) \]

Subtract

\[ y(t) = f(\phi(t)) + y(t) x(t) - f(\phi(t)) x(t) \]

\[ \frac{\partial f}{\partial x} x = \dot{\phi}(t) \]

Example: Precessing top