Newton-Euler Equations

From Dynamics of Systems of Particles to Rigid Body Dynamics
Outline

1. Newton’s 2\textsuperscript{nd} Law for a particle - recap
2. Newton’s 2\textsuperscript{nd} Law for a system of particles - recap
3. Center of mass, $F=ma$ for the system of particles
4. Angular momentum (relative to $O$ or the center of mass)
5. Newton’s 2\textsuperscript{nd} Law for Rigid Body Motion
   - $F=ma$
   - Rotational equation of motion
6. Principal axes and moments of inertia
7. Newton-Euler equations of motion
Newton’s Second Law for a Particle

Newton’s Second Law for a particle $P$
- Position vector $\mathbf{p}$ in an inertial frame $A$
- $\mathbf{F}$ is the force acting on the particle, mass $m$

\[ \mathbf{F} = m \frac{d}{dt} \mathbf{v}^P \]

Linear momentum in $A$
\[ \mathbf{L} = m \mathbf{v}^P \]

Angular Momentum of the system $S$ relative to $O$ in $A$
\[ \mathbf{H}_O^P = \sum_{i=1}^{N} \mathbf{p} \times m \frac{d}{dt} \mathbf{v}^P \]

Rate of change of angular and linear momentum in $A$
\[ \frac{d}{dt} \mathbf{L} = \ldots \]
\[ \frac{d}{dt} \mathbf{H}_O^P = \ldots \]
Newton’s Second Law for a System of Particles

Newton’s Second Law for a particle $P$
- Position vector $\mathbf{p}$ in an inertial frame $A$
- $\mathbf{F}$ is the force acting on the particle, mass $m$

$$\mathbf{F} = m \frac{d}{dt} \mathbf{v}^P$$

Linear momentum in $A$

$$\mathbf{L} = m \mathbf{v}^P$$

$$\mathbf{F} = \frac{d}{dt} \mathbf{L}$$

Extend to many particles
- Mass $m_i$ at $P_i$
- $F_i$ is the external force acting on $P_i$
- $F_{ij}$ is the force exerted by $P_j$ on $P_i$
Define \( \mathbf{r}_c = \frac{1}{m} \sum_{i=1,N} m_i \mathbf{p}_i \)

Newton's Second Law:

\[
\sum_{i=1}^{N} \left[ F_i + \sum_{j=1,j \neq i}^{N} F_{ij} \right] = \sum_{i=1}^{N} \left[ m_i \frac{d A_v P_i}{dt} \right]
\]

\[
\frac{d \mathbf{r}_c}{dt} = \frac{1}{m} \sum_{i=1,N} m_i \frac{d \mathbf{p}_i}{dt}
\]

\[
A_v C = \frac{d \mathbf{r}_c}{dt}
\]

\[
\sum_{i=1}^{N} F_i = m \frac{d A_v C}{dt}
\]
Newton’s Second Law for a System of Particles

The center of mass for a system of particles, $S$, accelerates in an inertial frame ($A$) as if it were a single particle with mass $m$ (equal to the total mass of the system) acted upon by a force equal to the net external force.

$$\text{Center of mass} \quad r_c = \frac{1}{m} \sum_{i=1}^{N} m_i \mathbf{p}_i$$

$$\mathbf{F} = \sum_{i=1}^{N} \mathbf{F}_i = m \frac{dA}{dt} \mathbf{v}^C$$

Conservation of Linear Momentum

The linear momentum of a system of particles stays constant in an inertial frame ($A$) if the net external force equals zero.
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Angular Momentum of a System of Particles

Angular Momentum of $P_i$ relative to $O$ in $A$

$$A \mathbf{H}_O^{P_i} = \mathbf{p}_i \times m_i \ A \frac{d \mathbf{p}_i}{dt}$$

Angular Momentum of $P_i$ relative to $C$ in $A$

$$A \mathbf{H}_C^{P_i} = \mathbf{r}_i \times m_i \ A \frac{d \mathbf{p}_i}{dt}$$

Note: $C$ is not fixed in $A$

Angular Momentum of the system $S$ relative to $O$ in $A$

$$A \mathbf{H}_O^S = \sum_{i=1}^{N} \mathbf{p}_i \times m_i \ A \frac{d \mathbf{p}_i}{dt}$$

Angular Momentum of the system $S$ relative to $C$ in $A$

$$A \mathbf{H}_C^S = \sum_{i=1}^{N} \mathbf{r}_i \times m_i \ A \frac{d \mathbf{p}_i}{dt}$$
Rate of Change of Angular Momentum relative to $O$

Angular Momentum of the system $S$ relative to $O$ in $A$

Resultant moment of all external forces acting on the system $S$ about (relative to) $O$

$$A \mathbf{H}^S_O = \sum_{i=1}^{N} \mathbf{p}_i \times m_i \dot{\mathbf{p}}_i$$

$$M^S_O = \sum_{i=1}^{N} \mathbf{p}_i \times \mathbf{F}_i$$

The rate of change of angular momentum of the system $S$ relative to $O$ in $A$ is equal to the resultant moment of all external forces acting on the system relative to $O$

$$\frac{A d A \mathbf{H}^S_O}{dt} = M^S_O$$

Notes:
1. $A$ is inertial; and
2. $O$ is fixed in $A$
Rate of Change of Angular Momentum relative to $C$

Angular Momentum of the system $S$ relative to $C$ in $A$

\[ A_\mathbf{H}_C^S = \sum_{i=1}^{N} \mathbf{r}_i \times m_i \dot{\mathbf{p}}_i \]

Resultant moment of all external forces acting on the system $S$ about (relative to) $C$

\[ \mathbf{M}_C^S = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i \]

The rate of change of angular momentum of the system $S$ relative to $C$ in $A$ is equal to the resultant moment of all external forces acting on the system relative to $C$

\[ \frac{A}{dt} \left( A_\mathbf{H}_C^S \right) = \frac{A}{dt} \sum_{i=1}^{N} (\mathbf{p}_i - \mathbf{r}_C) \times m_i \dot{\mathbf{p}}_i 
= \sum_{i=1}^{N} (\mathbf{p}_i - \mathbf{r}_C) \times m_i \dot{\mathbf{p}}_i + \sum_{i=1}^{N} \mathbf{r}_i \times m_i \ddot{\mathbf{p}}_i \]

\[ \frac{A}{dt} A_\mathbf{H}_C^S = \mathbf{M}_C^S \]
Principles of Conservation of Angular Momentum for a System of Particles

(1) \[ M^S_O = \sum_{i=1}^{N} p_i \times F_i = 0 \]

The angular momentum of \( S \) relative to \( O \) is constant in \( A \)

(2) \[ M^S_C = \sum_{i=1}^{N} r_i \times F_i = 0 \]

The angular momentum of \( S \) relative to \( C \) is constant in \( A \)
Examples

1. Particles $A$ and $B$, each of mass $m$, are attached to massless rods of length $l$ which are pivoted at $A$ and $O$ in the horizontal plane. Solve for the velocities as a function of positions, assuming $\theta$ and $\phi$ are zero at $t=0$, and the rate of change of $\theta$ and $\phi$ are known to be zero and $\omega_0$ respectively at $t=0$.

Answer

\[
\dot{\phi} = \omega_0 \sqrt{\frac{3 - 2\cos\phi}{1 + \sin^2\phi}}
\]

\[
\dot{\theta} = \frac{(1 - \cos\phi)\omega_0}{\sqrt{(1 + \sin^2\phi)(3 - 2\sin\phi)}}
\]
2. Each of the three balls has a mass $m$ and is welded to rigid angular frame of negligible mass. The assembly rests on a smooth horizontal surface. If a force $\mathbf{F}$ is applied to one bar as shown, determine (a) the acceleration of the point $O$ and the (b) the angular acceleration of the frame.

**Answer**

$$\mathbf{a}_O^a = \frac{1}{3m} \mathbf{F}$$

$$F_b = \frac{d}{dt}(-3mr^2 \omega)$$

$$\dot{\omega} = \frac{-F_b}{3mr^2}$$
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Angular Momentum for a System of Particles that are Rigidly Connected (relative to $O$)

$B$ is a reference frame that is attached to the rigid body of mass $m$

\[
\dot{\mathbf{r}}_i = \frac{A}{dt} \mathbf{r}_i = \frac{B}{dt} \mathbf{r}_i + A \omega_B \times \mathbf{r}_i
\]

\[
A \mathbf{H}_O^S = \sum_{i=1}^{N} \mathbf{r}_i \times m_i \mathbf{p}_i
\]

\[
= \sum_{i=1}^{N} \mathbf{p}_i \times m_i \frac{A}{dt} \left( \mathbf{r}_C + \mathbf{r}_i \right)
\]

\[
= \mathbf{r}_C \times m A \mathbf{v}_C + \sum_{i=1}^{N} \mathbf{r}_i \times m_i \left( A \omega_B \times \mathbf{r}_i \right)
\]

Angular momentum of a single particle of mass $m$ (equal to the sum of the $N$ masses) translating as if it were attached to the center of mass $C$.

Angular momentum of the rigid body due to its rotation.
Angular Momentum for a System of Particles that are Rigidly Connected (relative to $C$)

$B$ is a reference frame that is attached to the rigid body of mass $m$

\[
\dot{r}_i = \frac{A}{dt} \mathbf{r}_i = \frac{B}{dt} \mathbf{r}_i + \omega_B \times \mathbf{r}_i
\]

\[
A^H_C = \sum_{i=1}^{N} \mathbf{r}_i \times m_i \dot{p}_i
\]

\[
= \sum_{i=1}^{N} \mathbf{r}_i \times m_i \frac{A}{dt} (\mathbf{r}_C + \mathbf{r}_i)
\]

\[
= \sum_{i=1}^{N} \mathbf{r}_i \times m_i \left( \omega_B \times \mathbf{r}_i \right)
\]
Inertia Dyadic

\[ A^H_C = \sum_{i=1}^{N} r_i \times m_i \left( A^B \times r_i \right) \]

\[ I_C = \sum_{k,l=1}^{3} I_{kl} e_k e_l \]

\[ = \sum_{i=1}^{N} m_i \left[ (r_i \cdot r_i) U - r_i r_i \right] \]

\[ I_C \cdot A^B = \sum_{i=1,N} m_i \left[ (r_i \cdot r_i) U - r_i r_i \right] \cdot A^B \]

\[ = \sum_{i=1,N} m_i \left[ (r_i \cdot r_i) A^B - r_i \left( r_i \cdot A^B \right) \right] \]

\[ = \sum_{i=1,N} m_i r_i \times \left( A^B \times r_i \right) \]

\[ A^H_C = I_C \cdot A^B \]
Rotational equations of motion

The rate of change of angular momentum of the system $S$ relative to $C$ in $A$ is equal to the resultant moment of all external forces acting on the system relative to $C$

$$\frac{d}{dt} A H_C^S = M_C^S$$

What is the difficulty?

$$A H_C^S = I_C \cdot A \omega^B$$
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Principal Axes and Principal Moments

Principal axis of inertia

\( \mathbf{u} \) is a unit vector along a principal axis if \( \mathbf{I} \cdot \mathbf{u} \) is parallel to \( \mathbf{u} \).

There are 3 independent principal axes!

Principal moment of inertia

The moment of inertia with respect to a principal axis, \( \mathbf{u} \cdot \mathbf{I} \cdot \mathbf{u} \), is called a principal moment of inertia.

Physical interpretation

\( \mathbf{H}_O \) and \( \mathbf{A} \omega^B \) are not parallel!

\( \mathbf{H}_O \) and \( \mathbf{A} \omega^B \) are parallel!
Properties of the inertia matrix

With respect to any origin, and any reference triad!

1. The inertia matrix is **Symmetric**
2. The inertia matrix is **Positive Definite**
3. Moments of inertia are always positive
4. Moments of inertia are the least when $C$ is the reference point
5. It is always possible to find three distinct mutually orthogonal principal axes of inertia. [may not be unique]
6. If the reference triad is aligned with the principal axes, the products of inertia are zero
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Newton’s second law

\[ m \, {^A A \mathbf{a}}^C = \mathbf{R} \]

\[ \frac{^{A A \mathbf{H}}_C}{dt} = \mathbf{M}_C \]

\[ \frac{^{B A \mathbf{x}}_C}{dt} = \frac{^{B B \mathbf{x}}}{dt} + \omega \times \mathbf{H}_C = \mathbf{M}_C \]

If \( \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \) are along principal axes and

\[ ^A \mathbf{\omega}^B = \omega_1 \mathbf{b}_1 + \omega_2 \mathbf{b}_2 + \omega_3 \mathbf{b}_3 \]

\[ \mathbf{H}_C = I_{11} \omega_1 \mathbf{b}_1 + I_{22} \omega_2 \mathbf{b}_2 + I_{33} \omega_3 \mathbf{b}_3 \]

\[ \frac{^{B B \mathbf{H}}_C}{dt} = I_{11} \omega_1 \mathbf{b}_1 + I_{22} \omega_2 \mathbf{b}_2 + I_{33} \omega_3 \mathbf{b}_3 \]

Note \( C \) can be replaced by any point that is fixed in \( A \)
Euler’s Equations

\[
\frac{A}{dH_C}{dt} = M_C
\]

1

\[
\begin{bmatrix}
I_{11} & 0 & 0 \\
0 & I_{22} & 0 \\
0 & 0 & I_{33}
\end{bmatrix}
\begin{bmatrix}
\dot{\omega}_1 \\
\dot{\omega}_2 \\
\dot{\omega}_3
\end{bmatrix}
+ 
\begin{bmatrix}
0 & -\omega_3 & \omega_2 \\
\omega_3 & 0 & -\omega_1 \\
-\omega_2 & \omega_1 & 0
\end{bmatrix}
\begin{bmatrix}
I_{11} & 0 & 0 \\
0 & I_{22} & 0 \\
0 & 0 & I_{33}
\end{bmatrix}
\begin{bmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{bmatrix}
= 
\begin{bmatrix}
M_{C,1} \\
M_{C,2} \\
M_{C,3}
\end{bmatrix}
\]

2

Let \(b_1, b_2, b_3\), be along principal axes and

\[
A \vec{\omega}^B = \omega_1 b_1 + \omega_2 b_2 + \omega_3 b_3
\]

\[
\frac{B}{dH_C}{dt} + A \vec{\omega}^B \times \vec{H}_C = M_C
\]

\[
\frac{B}{dH_C}{dt} = I_{11}\dot{\omega}_1 b_1 + I_{22}\dot{\omega}_2 b_2 + I_{33}\dot{\omega}_3 b_3
\]
Example

The rectangular homogeneous plate is welded to the shaft and the assembly rotates about a vertical axis at uniform speed (ignore gravity). Find the forces and moments that the weld at $O$ must be subject to. What are the forces and moments that the bearings must support?

\[ I_O = \frac{m}{12} \begin{bmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & (a^2 + b^2) \end{bmatrix} \text{ with reference to } p_i \]

Problem

The rectangular homogeneous plate is welded to the shaft and the assembly rotates about a vertical axis at uniform speed (ignore gravity). Find the forces and moments that the weld at $O$ must be subject to. What are the forces and moments that the bearings must support?
Newton-Euler Equations (point $O$ fixed in $A$)

Newton’s equations

$$ m^A \mathbf{a}^C = \mathbf{R} $$

Euler’s equations

$$ \frac{A}{d\tau} \mathbf{H}_O = \mathbf{M}_O $$

$$ \mathbf{M}_O = \mathbf{M}_C + \mathbf{r}^{OC} \times \mathbf{R} $$

If $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$, are along principal axes and

$$ A^B \mathbf{\omega}^B = \omega_1 \mathbf{b}_1 + \omega_2 \mathbf{b}_2 + \omega_3 \mathbf{b}_3 $$

$$ \mathbf{H}_O = I_{O,11} \omega_1 \mathbf{b}_1 + I_{O,22} \omega_2 \mathbf{b}_2 + I_{O,33} \omega_3 \mathbf{b}_3 $$

$$ \frac{B}{d\tau} \mathbf{H}_O = I_{O,11} \dot{\omega}_1 \mathbf{b}_1 + I_{O,22} \dot{\omega}_2 \mathbf{b}_2 + I_{O,33} \dot{\omega}_3 \mathbf{b}_3 $$
Rotation of an axisymmetric body about a principal axis

\[
\frac{B}{dt} \frac{dH_C}{dt} + A \omega^B \times H_C = M_C
\]

Or

\[
\frac{B}{dt} \frac{dH_O}{dt} + A \omega^B \times H_O = M_O
\]

Suppose \( b_i \) are along principal axes, and

\[
A \omega^B = A \omega^F + F \omega^B,
\]

\[
F \omega^B = \Omega b_3
\]

And, suppose the rigid body is axisymmetric with \( b_3 \) as an axis of symmetry

Option 1

With respect to \( b_i \)

\[
H = I_{11}^B \omega_1^B b_1 + I_{22}^B \omega_2^B b_2 + I_{33}^B \omega_3^B b_3
\]

\[
\frac{A}{dt} \frac{dH_O}{dt} = \frac{B}{dt} \frac{dH_O}{dt} + \left( A \omega^B \right) \times H_O
\]

Option 2

With respect to \( f_i \)

\[
H = I_{11}^F \omega_1^F f_1 + I_{22}^F \omega_2^F f_2 + I_{33}^F \omega_3^F f_3
\]

\[
A \omega^B = \sum_i \omega_i^B b_i
\]

\[
= \sum_i \omega_i^F f_i
\]

Note: The angular velocity is \( A \omega^B \)

Which expression is simpler?
Example: Precession of a Spinning Top

Three simple angular velocities
- Spin $\psi$
- Precession $\dot{\phi}$
- Nutation $\dot{\theta}$

Three rotations
- Rotation about $a_3$ through $\phi$
- Rotation about $f_1$ through $\theta$
- Rotation about $f_3$ through $\psi$
Example (continued)

\[ A_{\omega}^B = \dot{\theta} \mathbf{f}_1 + \dot{\phi} \sin \theta \mathbf{f}_2 + \left( \dot{\phi} \cos \theta + \psi \right) \mathbf{f}_3 \]

\[ A_{\omega}^F = \dot{\theta} \mathbf{f}_1 + \dot{\phi} \sin \theta \mathbf{f}_2 + \dot{\phi} \cos \theta \mathbf{f}_3 \]
Example (continued)

\[ A \omega^B = \dot{\theta} f_1 + \dot{\phi} \sin \theta f_2 + (\phi \cos \theta + \psi) f_3 \]

\[ A \omega^F = \dot{\theta} f_1 + \dot{\phi} \sin \theta f_2 + \dot{\phi} \cos \theta f_3 \]

\[ \omega_1 = \dot{\theta} \]
\[ \omega_2 = \dot{\phi} \sin \theta \]
\[ \omega_3 = \dot{\phi} \cos \theta + \psi \]

Let $|OG| = l$

**Transverse moments of inertia**

**Axial moment of inertia**

\[
\begin{bmatrix}
I_{11} & 0 & 0 \\
0 & I_{22} & 0 \\
0 & 0 & I_{33}
\end{bmatrix}
\begin{bmatrix}
\ddot{\theta} \\
\dot{\phi} \sin \theta + \dot{\theta} \cos \theta \\
\dot{\phi} \cos \theta - \dot{\theta} \sin \theta + \psi
\end{bmatrix}
+ \begin{bmatrix}
\dot{\phi} \cos \theta & 0 & -\dot{\theta} \\
\dot{\phi} \cos \theta & 0 & -\dot{\theta} \\
-\dot{\phi} \sin \theta & \dot{\theta} & 0
\end{bmatrix}
\begin{bmatrix}
I_{11} & 0 & 0 \\
0 & I_{22} & 0 \\
0 & 0 & I_{33}
\end{bmatrix}
\begin{bmatrix}
\dot{\theta} \\
\dot{\phi} \sin \theta \\
\dot{\phi} \cos \theta + \psi
\end{bmatrix}
= \begin{bmatrix}
M_{O,1} \\
M_{O,2} \\
M_{O,3}
\end{bmatrix}
\]
A rate gyro produces a measurable nutation angle $\theta$ proportional to the precessional or yaw angular velocity. Assume the precession angular velocity is constant and is directed along the vertical axis, the linear spring holds the angle $\theta$ to small values and the linear damper keeps the angular rate small. Derive the equation of motion and show when accelerations and velocities are small, the angle $\theta$ is proportional to the yaw velocity.
Problem: Dynamics of a Thrown Football

A football is thrown horizontally at an angle $\theta$ with respect to the horizontal axis with a spin about its axial symmetry axis. The air drag can be assumed to act at a distance $e$ from the center of mass in the horizontal direction. Because the ball is not thrown perfectly there are small rotational velocity components in the transverse directions.

Write down equations of motion for the football and show that a sufficiently large spin velocity stabilizes the motion for $\theta=0$. 
Problem

A thin homogeneous disk of mass 800 grams and radius 100 mm rotates at a constant rate of $\omega_2 = 20$ rads/sec with respect to the arm $ABC$, which itself rotates at $\omega_1 = 10$ rads/sec in an inertial frame. For the position shown, determine the dynamic reactions at the bearings $D$ and $E$.

Problem

Two particles of mass $m$ are attached to a massless rod at an angle $\alpha$ as shown in the figure in such a way that the center of mass is on the horizontal shaft of length $2l$. Calculate the bearing reactions if the horizontal shaft rotates with a constant angular velocity $\omega$. 

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*University of Pennsylvania*