A rigid body $B$ has a *simple angular velocity* in $A$, when there exists a unit vector $\mathbf{k}$ whose orientation (as seen) in both $A$ and in $B$ is constant (independent of time).

Angular velocity of $B$ in $A$
- is along $\mathbf{a}_2$ as seen in $A$
- is along $\mathbf{b}_2$ as seen in $B$

*In each frame*, the angular velocity has a constant direction (magnitude may change)
$D\omega_A$, $A\omega_B$ and $C\omega_B$ are simple angular velocities.

However, $D\omega_C$ is not a simple angular velocity. The motion of $C$ relative to $D$ is such that there is no vector fixed in $D$ that also remains fixed in $C$. How about $D\omega_C$?
Addition Theorem for Angular Velocities

Let $A$, $B$, and $C$ be three rigid bodies.

The addition theorem for angular velocities states:

\[
A_\omega^C = A_\omega^B + B_\omega^C
\]

Proof

Let $r$ be fixed to $C$.

\[
\frac{A}{dt} \frac{dr}{dt} = \frac{B}{dt} \frac{dr}{dt} + A_\omega^B \times r
\]

\[
= \frac{C}{dt} \frac{dr}{dt} + B_\omega^C \times r + A_\omega^B \times r
\]

\[
= \frac{C}{dt} \frac{dr}{dt} + \left( B_\omega^C + A_\omega^B \right) \times r
\]

And,

\[
\frac{A}{dt} \frac{dr}{dt} = A_\omega^C \times r
\]
Angular velocities can be found by adding up “simple” angular velocities.

$D\omega^C$ is not a simple angular velocity. The motion of $C$ relative to $D$ is such that there is no vector fixed in $D$ that also remains fixed in $C$.

But,

$$D\omega^C = D\omega^A + A\omega^B + B\omega^C$$

$D\omega^A$, $A\omega^B$ and $C\omega^B$ are simple angular velocities.
A disk of radius $r$ is mounted on an axle $OG$. The disk rotates counterclockwise at the constant rate of $\omega_1$ about $OG$.

Determine the angular velocity of the disk in an inertial frame.

Determine the velocity and acceleration of the point $P$ in an inertial frame.
Example

The rolling (and sliding) disk on a horizontal plane

A circular disk \( C \) of radius \( R \) is in contact with a horizontal plane (not shown in the figure) at the point \( P \). The point \( P \) is attached to the disk. The plane is the \( x-y \) plane. It is rigidly attached to the earth. The standard reference triad \( \mathbf{b}_i \) is chosen so that \( \mathbf{b}_1 \) is along the direction of progression of the disk (parallel to the tangent to the disk at \( P \)), \( \mathbf{b}_2 \) is parallel to the plane of the disk, and \( \mathbf{b}_3 \) is normal to the disk. Note that this triad is not fixed to the disk. Call the earth-fixed reference frame \( A \) and choose the standard reference triad \( \mathbf{a}_x \), \( \mathbf{a}_y \), and \( \mathbf{a}_z \) in an obvious fashion along the \( x \), \( y \), and \( z \) axes shown in the figure.
Reference Triads

- Rotate triad \( A \) about \( z \) through \( q_1 \) followed by rotation about \( x \) by 90 deg to get \( E \)
- Rotate triad \( E \) about \(-x\) through \( q_2 \) to get \( B \)
- Rotate triad \( B \) about \( z \) through \( q_3 \) to get \( C \) (not shown)

Imagine \( E \) to be a virtual body that is attached to \( Q \)

Locus of the point of contact \( Q \) on the plane \( A \)

Imagine \( B \) to be a virtual body that is attached to \( C^* \)
Two coordinate transformations

- $a_i$ in terms of $e_i$
- $e_i$ in terms of $b_i$
Reference Triads

- Rotate triad $A$ about $z$ through $q_1$ followed by rotation about $x$ by 90 deg to get $E$
- Rotate triad $E$ about $-x$ through $q_2$ to get $B$
- Rotate triad $B$ about $z$ through $q_3$ to get $C$ (not shown)

Imagine $E$ to be a virtual body that is attached to $Q$

Locus of the point of contact $Q$ on the plane $A$

Imagine $B$ to be a virtual body that is attached to $C^*$

University of Pennsylvania
Transformations

Two coordinate transformations

- \( \mathbf{a}_i \) in terms of \( \mathbf{e}_i \)
- \( \mathbf{e}_i \) in terms of \( \mathbf{b}_i \)

\[
\begin{align*}
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos q_2 & -\sin q_2 \\
0 & \sin q_2 & \cos q_2
\end{bmatrix}
\begin{bmatrix}
\cos q_1 & \sin q_1 & 0 \\
0 & 0 & 1 \\
\sin q_1 & -\cos q_1 & 0
\end{bmatrix}
&= \\
\begin{bmatrix}
\cos q_1 & \sin q_1 & 0 \\
-\sin q_1 & \sin q_2 & \cos q_1 \sin q_2 & \cos q_2 \\
\sin q_1 \cos q_2 & -\cos q_1 \cos q_2 & \sin q_2
\end{bmatrix}
\end{align*}
\]
Angular Velocity: Components

\[ A\omega^C = u_1 \mathbf{b}_1 + u_2 \mathbf{b}_2 + u_3 \mathbf{b}_3 \]

- \( u_i \) are the components of the angular velocity of the disk with respect to the reference triad \( B \)

\[ A\omega^C = u_x \mathbf{a}_x + u_y \mathbf{a}_y + u_z \mathbf{a}_z \]

- \( u_\alpha \) are the components of the angular velocity of the disk with respect to the reference triad \( A \)

\[
\begin{bmatrix}
    u_1 \\
    u_2 \\
    u_3 \\
    u_4 \\
    u_5
\end{bmatrix} = \begin{bmatrix}
    \dot{q}_1 \\
    \dot{q}_2 \\
    \dot{q}_3 \\
    \dot{q}_4 \\
    \dot{q}_5
\end{bmatrix} = X
\]

\[
\begin{bmatrix}
    \dot{q}_1 \\
    \dot{q}_2 \\
    \dot{q}_3 \\
    \dot{q}_4 \\
    \dot{q}_5
\end{bmatrix} = \begin{bmatrix}
    u_1 \\
    u_2 \\
    u_3 \\
    u_4 \\
    u_5
\end{bmatrix} = Y
\]