Kinematics

Velocity and Acceleration Analysis
General Approach to Analyzing Multi-Body System

1. Points common to adjacent bodies

2. Pair of points fixed to the same body

3. Pair of points on adjacent bodies whose relative motions can be easily described

4. Write equations relating pairs of points either on same or adjacent bodies.
Notation Revisited…

- Points $P$ fixed to (in) $B$ and point $Q$ moving in $Q$
- $O$ is a point fixed in $A$
- Position vectors for $P$ and $Q$ in $A$ are denoted by $p$ and $q$
- Velocities for $P$ and $Q$ in $A$
  \[
  A_v^P = \frac{A}{dt} \frac{dp}{dt}, \quad A_v^Q = \frac{A}{dt} \frac{dq}{dt}
  \]
- Accelerations for $P$ and $Q$ in $A$
  \[
  A_a^P = \frac{A}{dt} \frac{A_v^P}{dt}, \quad A_a^Q = \frac{A}{dt} \frac{A_v^Q}{dt}
  \]

Notice consistency in leading superscripts!
Example

Two bugs are on a rotating turntable with angular speed $\omega$. The one at P is stationary on the turntable while the one at Q is moving radially away from P at a uniform speed.

Let $OP$ be $r_0$, a constant. Let the magnitude of $PQ$ be $r$. $r$ and $\omega$ are variables. Let the rate of change of $r$ be $v$.

Find the acceleration of both bugs when $\theta = 0$. 
Differentiation of vectors

2. Vector not fixed to $B$

\[
\frac{A}{dt} \frac{d\mathbf{r}}{d\mathbf{t}} = \frac{d\mathbf{r}_1}{dt} \mathbf{b}_1 + \frac{d\mathbf{r}_2}{dt} \mathbf{b}_2 + \frac{d\mathbf{r}_3}{dt} \mathbf{b}_3 + \mathbf{r}_1 \frac{A}{dt} \mathbf{b}_1 + \mathbf{r}_2 \frac{A}{dt} \mathbf{b}_2 + \mathbf{r}_3 \frac{A}{dt} \mathbf{b}_3
\]

\[
= \frac{B}{dt} \mathbf{b}_1 + \mathbf{r}_1 A \omega^B \times \mathbf{b}_1 + \mathbf{r}_2 A \omega^B \times \mathbf{b}_2 + \mathbf{r}_3 A \omega^B \times \mathbf{b}_3
\]

\[
= \frac{B}{dt} \mathbf{r} + A \omega^B \times \mathbf{r}
\]

\[
\frac{A}{dt} \frac{d\mathbf{r}}{d\mathbf{t}} = \frac{B}{dt} \mathbf{r} + A \omega^B \times \mathbf{r}
\]

$r$ can be any vector
Relationship between Velocities and Accelerations in $A$ of Two Points fixed to $B$

- Points $P$ and $Q$ fixed to (in) $B$
- $O$ is a point fixed in $A$
- Position vectors for $P$ and $Q$ in $A$ are denoted by $\mathbf{p}$ and $\mathbf{q}$
- Velocities for $P$ and $Q$ in $A$
  \[ A\mathbf{v}_P = \frac{A\mathbf{d}\mathbf{p}}{dt}, \quad A\mathbf{v}_Q = \frac{A\mathbf{d}\mathbf{q}}{dt} \]
- Accelerations for $P$ and $Q$ in $A$
  \[ A\mathbf{a}_P = \frac{A\mathbf{d}A\mathbf{v}_P}{dt}, \quad A\mathbf{a}_Q = \frac{A\mathbf{d}A\mathbf{v}_Q}{dt} \]

Notice consistency in leading superscripts!
Velocities of $P$ and $Q$ (both fixed in $A$)

- Triangle law of vector addition for points $P$ and $Q$
  \[ q = p + r \]

- Differentiate both sides
  \[ \frac{A}{dt} d q = \frac{A}{dt} d p + \frac{A}{dt} d r \]

- Substitute definitions of velocities
  \[ A v_Q = A v_P + \left( B \frac{dr}{dt} + A \omega^B \times r \right) \]

- Velocities for $P$ and $Q$ in $A$
  \[ A v_Q = A v_P + A \omega^B \times r \]
Angular Acceleration

The *angular acceleration of B in A*, denoted by \( ^A \alpha^B \), is defined as the first time-derivative in \( A \) of the angular velocity of \( B \) in \( A \):

\[
^A \alpha^B = \frac{d}{dt} \left( ^A \omega^B \right)
\]

*Notice consistency in leading superscripts!*
Accelerations of $P$ and $Q$

- Velocities for $P$ and $Q$ in $A$
  \[ A\vec{v}^Q = A\vec{v}^P + A\vec{\omega}^B \times \vec{r} \]

- Differentiate both sides
  \[ \frac{A}{dt} \left( A\vec{v}^Q \right) = \frac{A}{dt} \left( A\vec{v}^P \right) + \frac{A}{dt} \left( A\vec{\omega}^B \times \vec{r} \right) \]

  \[ A\vec{a}^Q = A\vec{a}^P + A\frac{d\vec{v}^P}{dt} + A\frac{dA\vec{\omega}^B}{dt} \times \vec{r} \]

- Accelerations for $P$ and $Q$ in $A$
  \[ A\vec{a}^Q = A\vec{a}^P + A\vec{a}_1 \times \vec{r} + A\vec{a}_3 \times \left( A\vec{\omega}^B \times \vec{r} \right) \]

  - tangential acceleration
  - centripetal (normal) acceleration
Relationship between Velocities and Accelerations in \( A \) of Points described in \( B \)

- Point \( P \) fixed to (in) \( B \)
- Point \( Q \) moving in \( B \) (but easily described in \( B \))
Velocities of $P$ and $Q$

- Triangle law of vector addition for points $P$ and $Q$
  \[ \mathbf{q} = \mathbf{p} + \mathbf{r} \]
- Differentiate both sides
  \[ \frac{d}{dt} \mathbf{Aq} = \frac{d}{dt} \mathbf{Ap} + \frac{d}{dt} \mathbf{Ar} \]
- Substitute definitions of velocities
  \[ \mathbf{A} \mathbf{v}_Q = \mathbf{A} \mathbf{v}_P + \left( \frac{d}{dt} \mathbf{A} \mathbf{r} + \mathbf{A} \mathbf{\omega}_B \times \mathbf{r} \right) \]
- Velocities for $P$ and $Q$ in $A$
  \[ \mathbf{A} \mathbf{v}_Q = \mathbf{A} \mathbf{v}_P + \mathbf{B} \mathbf{v}_Q + \mathbf{A} \mathbf{\omega}_B \times \mathbf{r} \]
Velocity and Acceleration of $Q$ in $B$

- Point $Q$ moving in $B$ has position vector $s$ in $B$
  \[ s = \overline{QQ} \]

- Velocity
  \[ B \mathbf{v}_Q = \frac{B}{dt} \]

- Acceleration
  \[ B \mathbf{a}_Q = \frac{B}{dt} \left( B \mathbf{v}_Q \right)' \]

Note: $s$ may be 0, but $\mathbf{v}$, $\mathbf{a}$ may not be!
Acceleration of $P$ and $Q$

- Velocities for $P$ and $Q$ in $A$
  $$A\mathbf{v}^Q = A\mathbf{v}^P + B\mathbf{v}^Q + A\omega^B \times \mathbf{r}$$

- Differentiate in $A$
  $$A\mathbf{a}^Q = \frac{A}{dt}(A\mathbf{v}^Q) = \frac{A}{dt}(A\mathbf{v}^P) + \frac{A}{dt}(B\mathbf{v}^Q) + \frac{A}{dt}(A\omega^B \times \mathbf{r})$$
  $$\left\{ A\mathbf{a}^P \right\}$$
  $$\left\{ \frac{A}{dt}(A\omega^B) \times \mathbf{r} + A\omega^B \times \left( \frac{B}{dt}\mathbf{r} + A\omega^B \times \mathbf{r} \right) \right\}$$
  $$\left\{ \frac{B}{dt}(B\mathbf{v}^Q) + A\omega^B \times B\mathbf{v}^Q \right\}$$
Acceleration of $P$ and $Q$

\[ \mathbf{a}^{Q} = \mathbf{a}^{P} + \mathbf{a}^{Q} + A_{\alpha}B \times \mathbf{r} + A_{\alpha}B \times \left( A_{\alpha}B \times \mathbf{r} \right) + 2A_{\omega}B \times B_{Q} \mathbf{v}^{Q} \]

- Tangential acceleration
- Centripetal (normal) acceleration
- Coriolis acceleration

Special case: $\mathbf{r} = 0$

\[ \mathbf{a}^{Q} = \mathbf{a}^{Q} + B_{Q} \mathbf{v}^{Q} \]
Example

Two bugs are on a rotating turntable. The one at P is stationary on the turntable while the one at Q is moving radially away from P at a uniform speed. (the rate of change of \( r \), the magnitude of \( PQ \), is constant).

Choose \( \theta = 0 \).

Let both \( P \) and \( Q \) be instantaneously coincident, but \( Q \) is moving radially outward.
Example

The motor housing and its bracket rotate about the Z-axis at the constant rate $\Omega = 3 \text{ rad/s}$. The motor shaft and disk have a constant angular velocity of spin $\varpi = 8 \text{ rad/s}$ with respect to the motor housing in the direction shown. If $\gamma$ is constant at $30^\circ$, determine the velocity and acceleration of point $A$ at the top of the disk and the angular acceleration $\alpha$ of the disk.

$I$ – inertial frame
$E$ – bracket
$M$ – motor housing
$D$ – disk
$XYZ$ – fixed to the bracket
(unit vectors – $e_i$)
$xyz$ – fixed to the motor (unit
vectors – $m_i$)
A disk of radius $r$ is mounted on an axle $OG$ of negligible mass. The disk rotates counter-clockwise rolling on the flat plate (fixed to the inertial frame) at the constant rate $\omega_1$ about $OG$.

Determine the angular velocity and angular acceleration of the disk in an inertial frame.

Find the acceleration of the contact point $P$. 
A disk of radius $r$ is mounted on an axle $OG$ of negligible mass. The disk rotates counter-clockwise rolling on the flat plate (fixed to the inertial frame) at the constant rate $\omega_1$ about $OG$.

Determine the angular velocity and angular acceleration of the disk in an inertial frame.

Find the acceleration of the contact point $P$. 
Solution
Example

A gyroscope consists of a rotor with its mass center fixed in space but which can spin freely about its geometric axis and assume any orientation.

- From a reference position with gimbals and a reference diameter of the rotor aligned, the gyroscope may be brought to any orientation through a succession of three steps:
  1) rotation of outer gimbal through $\varphi$ about $AA'$,
  2) rotation of inner gimbal through $\theta$ about $BB'$,
  3) rotation of the rotor through $\psi$ about $CC'$.

$\dot{\varphi} = $ rate of precession

$\dot{\theta} = $ rate of nutation

$\dot{\psi} = $ rate of spin

Determine the angular velocity and angular acceleration of the rotor in an inertial frame.
Example: Gyroscope

\[
A \omega^D = -\dot{\phi} \sin \theta c_1 + \dot{\theta} c_2 + (\phi \cos \theta + \dot{\psi}) c_3
\]