Position, Velocity and Angular Velocity Vectors
Position, Velocity and Acceleration Vectors

\( \mathbf{p} \) is a position vector of \( P \) in \( A \)
- Emanates from a point fixed to \( A \)
- Ends up at \( P \)

\( A \mathbf{v}^P \) is the velocity of \( P \) in \( A \)
\[
A \mathbf{v}^P = \frac{A \mathbf{d}\mathbf{p}}{dt}
\]

\( A \mathbf{a}^P \) is the acceleration of \( P \) in \( A \)
\[
A \mathbf{a}^P = \frac{A \mathbf{d}(A \mathbf{v}^P)}{dt}
\]

What if a different position vector were chosen?
Velocity of \( P \) in \( A \) is independent of choice of “origin” in \( A \)!

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- Emanates from a point fixed to \( A \)
- Ends up at \( P \)

\( A \mathbf{v}^P \) is the velocity of \( P \) in \( A \)

\[
A \mathbf{v}^P = \frac{A \mathbf{d} \mathbf{p}'}{dt} = \frac{A \mathbf{d}(\mathbf{p} + \mathbf{r})}{dt}
\]
A robotic arm is a system of rigid bodies (reference frames) \( B, C, \) and \( D \). \( A \) is the inertial or the laboratory reference frame that is considered fixed.

Example 1

1. Are the following equal?

\[
\frac{\partial \mathbf{p}}{\partial q_2} \quad \frac{A \partial \mathbf{p}}{\partial q_2} \quad \frac{B \partial \mathbf{p}}{\partial q_2} \quad \frac{C \partial \mathbf{p}}{\partial q_2}
\]

2. If motor (joint) rates are given, calculate \( \frac{A d\mathbf{p}}{dt} \)

\[
\frac{A d\mathbf{p}}{dt} = \frac{A \partial \mathbf{p}}{\partial q_1} \dot{q}_1 + \frac{A \partial \mathbf{p}}{\partial q_2} \dot{q}_2 + \frac{A \partial \mathbf{p}}{\partial q_3} \dot{q}_3
\]

3. Find the velocity of \( Q \) in \( A \)
The angular velocity of B in A, denoted by $^A \omega^B$, is defined as:

$$^A \omega^B = b_1 \left( \frac{A}{dt} b_2 \cdot b_3 \right) + b_2 \left( \frac{A}{dt} b_3 \cdot b_1 \right) + b_3 \left( \frac{A}{dt} b_1 \cdot b_2 \right)$$

- Defined in terms of a reference triad attached to B
- *Independent* of reference triad attached to A
- Generalizes to three dimensions
- Yields simple results for derivatives of vectors

Example

$$b_1 = a_1 \cos \theta + a_2 \sin \theta$$

$$b_2 = a_2 \cos \theta - a_1 \sin \theta$$
Alternative Formulation: Angular Velocity

$B$ rotates as seen by an observer attached to $A$.

Consider the position vector, $p$, in $A$, of a point $P$ fixed to $B$.

**What is the velocity of $P$ in $A$ in terms of components with respect to the SRT $a_i$?**

$$
\begin{align*}
\left[ A \mathbf{v}^P \right]^A &= \frac{d}{dt} [p]^A \\
\frac{d}{dt}[p]^A &= \frac{d}{dt} \left( A R_B [p]^B \right)
\end{align*}
$$
Angular Velocity and Rotation Matrix

\[ [A R_B] = \begin{bmatrix} b_1 \cdot a_1 & b_2 \cdot a_1 & b_3 \cdot a_1 \\ b_1 \cdot a_2 & b_2 \cdot a_2 & b_3 \cdot a_2 \\ b_1 \cdot a_3 & b_2 \cdot a_3 & b_3 \cdot a_3 \end{bmatrix} \]

Define (\(a db_i/dt\) = \(\dot{b}_i\))

\[ \dot{[A R_B]} = \begin{bmatrix} \dot{b}_1 \cdot a_1 & \dot{b}_2 \cdot a_1 & \dot{b}_3 \cdot a_1 \\ \dot{b}_1 \cdot a_2 & \dot{b}_2 \cdot a_2 & \dot{b}_3 \cdot a_2 \\ \dot{b}_1 \cdot a_3 & \dot{b}_2 \cdot a_3 & \dot{b}_3 \cdot a_3 \end{bmatrix} \]

\[ [A R_B]^T [A \dot{R}_B] = \begin{bmatrix} b_1 \cdot a_1 & b_2 \cdot a_1 & b_3 \cdot a_1 \\ b_1 \cdot a_2 & b_2 \cdot a_2 & b_3 \cdot a_2 \\ b_1 \cdot a_3 & b_2 \cdot a_3 & b_3 \cdot a_3 \end{bmatrix}^T \begin{bmatrix} \dot{b}_1 \cdot a_1 & \dot{b}_2 \cdot a_1 & \dot{b}_3 \cdot a_1 \\ \dot{b}_1 \cdot a_2 & \dot{b}_2 \cdot a_2 & \dot{b}_3 \cdot a_2 \\ \dot{b}_1 \cdot a_3 & \dot{b}_2 \cdot a_3 & \dot{b}_3 \cdot a_3 \end{bmatrix} \]

\[ [A R_B]^T [A \dot{R}_B] = \begin{bmatrix} b_1 \cdot \dot{b}_1 & b_1 \cdot \dot{b}_2 & b_1 \cdot \dot{b}_3 \\ b_2 \cdot \dot{b}_1 & b_2 \cdot \dot{b}_2 & b_2 \cdot \dot{b}_3 \\ b_3 \cdot \dot{b}_1 & b_3 \cdot \dot{b}_2 & b_3 \cdot \dot{b}_3 \end{bmatrix} \]

\[ [A R_B]^T [A \dot{R}_B] = \begin{bmatrix} 0 & -b_2 \cdot \dot{b}_1 & b_1 \cdot \dot{b}_3 \\ b_2 \cdot \dot{b}_1 & 0 & -b_3 \cdot \dot{b}_2 \\ -b_1 \cdot \dot{b}_3 & b_3 \cdot \dot{b}_2 & 0 \end{bmatrix} \]

Compare

\[ A \omega^B = b_1 (\dot{b}_2 \cdot b_3) + b_2 (\dot{b}_3 \cdot b_1) + b_3 (\dot{b}_1 \cdot b_2) \]

The angular velocity vector can be obtained by differentiating the rotation matrix!
Recall every $3 \times 1$ vector $\mathbf{a}$ has a $3 \times 3$ skew symmetric matrix counterpart $\mathbf{A}$ or $\hat{\mathbf{a}}$

\[
\begin{bmatrix}
    a_1 \\
    a_2 \\
    a_3
\end{bmatrix} \times \begin{bmatrix}
    b_1 \\
    b_2 \\
    b_3
\end{bmatrix} = \begin{bmatrix}
    a_2 b_3 - a_3 b_2 \\
    a_3 b_1 - a_1 b_3 \\
    -a_2 b_1 + a_1 b_2
\end{bmatrix}
\]

\[
= \begin{bmatrix}
    0 & -a_3 & a_2 \\
    a_3 & 0 & -a_1 \\
    -a_2 & a_1 & 0
\end{bmatrix} \begin{bmatrix}
    b_1 \\
    b_2 \\
    b_3
\end{bmatrix}
\]

For any vector $\mathbf{b}$

\[
\mathbf{a} \times \mathbf{b} = \mathbf{A} \mathbf{b}
\]
Angular velocity components in $A$ and $B$

$A$ - Inertial

$B$ - Body fixed frame

\[
\frac{d}{dt} [p]^A = \frac{d}{dt} (A R_B [p]^B) \quad \downarrow \quad = A \dot{R}_B [p]^B
\]

Components of $^A v^P$ in $A$

Pre multiply by $^B R_A$ to get components of $^A v^P$ in $B$

\[
\]

Components of position vector of $P$ in $A$ in $B$
Differentiation of vectors

1. Vector fixed to $B$

$$\mathbf{r} = r_1 \mathbf{b}_1 + r_2 \mathbf{b}_2 + r_3 \mathbf{b}_3$$

$$\frac{A \, d\mathbf{r}}{dt} = r_1 \frac{A \, d\mathbf{b}_1}{dt} + r_2 \frac{A \, d\mathbf{b}_2}{dt} + r_3 \frac{A \, d\mathbf{b}_3}{dt}$$

Composition and Projection rule

$$\frac{A \, d\mathbf{b}_1}{dt} = \left( \frac{A \, d\mathbf{b}_1}{dt} \cdot \mathbf{b}_1 \right) \mathbf{b}_1 + \left( \frac{A \, d\mathbf{b}_1}{dt} \cdot \mathbf{b}_2 \right) \mathbf{b}_2 + \left( \frac{A \, d\mathbf{b}_1}{dt} \cdot \mathbf{b}_3 \right) \mathbf{b}_3$$

$$-\left( \mathbf{b}_1 \cdot \frac{A \, d\mathbf{b}_3}{dt} \right)$$
Differentiation of vectors (cont’d)

\[ A_\omega B \times b_1 = \left[ b_1 \left( \frac{A\,db_2}{dt}.b_3 \right) + b_2 \left( \frac{A\,db_3}{dt}.b_1 \right) + b_3 \left( \frac{A\,db_1}{dt}.b_2 \right) \right] \times b_1 \]

\[ \frac{A\,db_1}{dt} = \left( \frac{A\,db_1}{dt}.b_2 \right) b_2 - \left( \frac{A\,db_3}{dt}.b_1 \right) b_3 \]

Important Result 1

\[ \frac{A\,db_i}{dt} = A_\omega B \times b_i \]

Important Result 2

\[ \frac{A\,dr}{dt} = r_1 \frac{A\,db_1}{dt} + r_2 \frac{A\,db_2}{dt} + r_3 \frac{A\,db_3}{dt} \]

\[ = r_1 A_\omega B \times b_1 + r_2 A_\omega B \times b_2 + r_3 A_\omega B \times b_3 \]

\[ = A_\omega B \times r \]
Example 2


\[ \frac{B}{dt} \frac{dr}{dt} = ? \]

2. Assuming $B$ is fixed to $A$

\[ \frac{A}{dt} \frac{dr}{dt} = ? \]
Velocity of $P$ (attached to $B$) in $A$ when $A$ and $B$ have a common point $O$

Choose $\mathbf{p}$ to be a position vector of $P$ in $A$

$$\mathbf{v}_P^A = \frac{d\mathbf{p}}{dt} = \omega^B \times \mathbf{p}$$
Differentiation of vectors

2. Vector not fixed to $B$

\[
\frac{A}{dt} \mathbf{r} = \frac{d\mathbf{r}_1}{dt} \mathbf{b}_1 + \frac{d\mathbf{r}_2}{dt} \mathbf{b}_2 + \frac{d\mathbf{r}_3}{dt} \mathbf{b}_3 + r_1 \frac{A}{dt} \mathbf{b}_1 + r_2 \frac{A}{dt} \mathbf{b}_2 + r_3 \frac{A}{dt} \mathbf{b}_3
\]

\[
= \frac{B}{dt} \mathbf{r} + A \omega^B \times \mathbf{r}
\]

\[
\frac{A}{dt} \mathbf{r} = \frac{B}{dt} \mathbf{r} + A \omega^B \times \mathbf{r}
\]

$r$ can be any vector
A rigid body $B$ has a *simple angular velocity* in $A$, when there exists a unit vector $\mathbf{k}$ whose orientation (as seen) in both $A$ and in $B$ is constant (independent of time).

Angular velocity of $B$ in $A$
- is along $\mathbf{a}_2$ as seen in $A$
- is along $\mathbf{b}_2$ as seen in $B$

*In each frame*, the angular velocity has a constant direction (magnitude may change)
Simple Angular Velocity

$D \omega_A$, $A \omega_B$ and $C \omega_B$ are simple angular velocities.

However, $D \omega_C$ is not a simple angular velocity. The motion of $C$ relative to $D$ is such that there is no vector fixed in $D$ that also remains fixed in $C$.

How about $D \omega_C$?
Addition Theorem for Angular Velocities

Let $A$, $B$, and $C$ be three rigid bodies.

The addition theorem for angular velocities states:

$$A \omega^C = A \omega^B + B \omega^C$$

Proof

Let $\mathbf{r}$ be fixed to $C$.

$$\frac{A \, d\mathbf{r}}{dt} = \frac{B \, d\mathbf{r}}{dt} + A \omega^B \times \mathbf{r}$$

$$= \frac{C \, d\mathbf{r}}{dt} + B \omega^C \times \mathbf{r} + A \omega^B \times \mathbf{r}$$

$$= \frac{C \, d\mathbf{r}}{dt} + \left( B \omega^C + A \omega^B \right) \times \mathbf{r}$$

$$= \left( B \omega^C + A \omega^B \right) \times \mathbf{r}$$

And,

$$\frac{A \, d\mathbf{r}}{dt} = A \omega^C \times \mathbf{r}$$
Angular velocities can be found by adding up “simple” angular velocities

\[ D\vec{\omega}^C \] is not a simple angular velocity. The motion of \( C \) relative to \( D \) is such that there is no vector fixed in \( D \) that also remains fixed in \( C \).

But,

\[ D\vec{\omega}^C = D\vec{\omega}^A + A\vec{\omega}^B + B\vec{\omega}^C \]

\( D\vec{\omega}^A, A\vec{\omega}^B \) and \( C\vec{\omega}^B \) are simple angular velocities.
A disk of radius $r$ is mounted on an axle $OG$. The disk rotates counter-clockwise at the constant rate of $\omega_1$ about $OG$.

Determine the angular velocity of the disk in an inertial frame if the disk rolls on the plane.

Determine the velocity and acceleration of the point $P$ in an inertial frame.
Example 3 (continued)
The rolling (and sliding) disk on a horizontal plane

A circular disk $C$ of radius $R$ is in contact with a horizontal plane (not shown in the figure) at the point $P$. The point $P$ is attached to the disk. The plane is the $x$-$y$ plane. It is rigidly attached to the earth. The standard reference triad $\mathbf{b}_i$ is chosen so that $\mathbf{b}_1$ is along the direction of progression of the disk (parallel to the tangent to the disk at $P$), $\mathbf{b}_2$ is parallel to the plane of the disk, and $\mathbf{b}_3$ is normal to the disk. Note that this triad is not fixed to the disk. Call the earth-fixed reference frame $A$ and choose the standard reference triad $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ in an obvious fashion along the $x$, $y$, and $z$ axes shown in the figure.
Reference Triads

- Rotate triad $A$ about $z$ through $q_1$ followed by rotation about $x$ by 90 deg to get $E$
- Rotate triad $E$ about $-x$ through $q_2$ to get $B$
- Rotate triad $B$ about $z$ through $q_3$ to get $C$ (not shown)

Imagine $E$ to be a virtual body that is attached to $Q$

Imagine $B$ to be a virtual body that is attached to $C^*$

Locus of the point of contact $Q$ on the plane $A$
Find the transformation

- $a_i$ in terms of $b_i$
Transformations

Two coordinate transformations

- \( \mathbf{a}_i \) in terms of \( \mathbf{e}_i \)

\[ E \mathbf{R}_A \]

- \( \mathbf{e}_i \) in terms of \( \mathbf{b}_i \)

\[ B \mathbf{R}_E \]

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos q_2 & -\sin q_2 \\
0 & \sin q_2 & \cos q_2
\end{bmatrix}
\begin{bmatrix}
\cos q_1 & \sin q_1 & 0 \\
0 & 0 & 1 \\
\sin q_1 & -\cos q_1 & 0
\end{bmatrix}
= \begin{bmatrix}
\cos q_1 & \sin q_1 & 0 \\
-sin q_1 \sin q_2 & \cos q_1 \sin q_2 & \cos q_2 \\
\sin q_1 \cos q_2 & -\cos q_1 \cos q_2 & \sin q_2
\end{bmatrix}
\]
Angular Velocity: Components

\[ A \omega^C = u_1 \mathbf{b}_1 + u_2 \mathbf{b}_2 + u_3 \mathbf{b}_3 \]

- \( u_i \) are the components of the angular velocity of the disk with respect to the reference triad \( B \)

\[ \dot{u}_i \]

\[ A \omega^C = u_x \mathbf{a}_x + u_y \mathbf{a}_y + u_z \mathbf{a}_z \]

- \( u_\alpha \) are the components of the angular velocity of the disk with respect to the reference triad \( A \)

\[ \dot{u}_\alpha \]