Lagrange’s Equations and Kane’s Formulation of Lagrange’s Equations
Kane’s Equations for Holonomic Systems

The motion of a nonholonomic system with $N$ particles and $n$ speeds, all independent, is governed by $n$ equations of motion:

$$Q_j + \dot{Q}_j^* = 0, \quad j = 1, 2, \ldots, n$$

$$Q_j = \sum_{i=1}^{N} [F_{ij}^{(a)} \cdot v_{ij}^{P}]$$

$$\dot{Q}_j^* = \sum_{i=1}^{N} [-m_i a_i \cdot v_{ij}^{P}]$$

**Generalized (active) force**  **Generalized (active) inertial force**

**Advantage**

- Principle of virtual work (Galileo, Bernoulli)
  - Ignore reaction forces and focus on the forces and moments that do work
- D’Alembert’s principle
  - Can incorporate inertial forces
- Kane’s equation: One equation of motion for each degree of freedom
Key identity

\[
\mathbf{a} \cdot \frac{\partial \mathbf{v}}{\partial \dot{q}_j} = \frac{d}{dt} \left( \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial \dot{q}_j} \right) - \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial q_j} \quad \Rightarrow \quad m_i \mathbf{a}_i \cdot \mathbf{v}_j = \frac{1}{2} \sum_{k=1}^{n} \left[ \frac{d}{dt} \left( \frac{\partial m_i (\mathbf{v}_i \cdot \mathbf{v}_i)}{\partial \dot{q}_j} \right) - \frac{\partial m_i (\mathbf{v}_i \cdot \mathbf{v}_i)}{\partial q_j} \right]
\]

Define Kinetic Energy of \( P_i \)

\[
T_i = \frac{1}{2} m_i \left( \mathbf{v}_i \cdot \mathbf{v}_i \right)
\]

\[
Q_j^* = \sum_{i=1}^{N} \left[ -m_i \mathbf{a}_i \cdot \mathbf{v}_j \right]
\]

Define Kinetic Energy of the system

\[
T = \sum_{i=1}^{N} \frac{1}{2} m_i \left( \mathbf{v}_i \cdot \mathbf{v}_i \right)
\]

\[
Q_j^* = -\sum_{i=1}^{N} m_i \mathbf{a}_i \cdot \mathbf{v}_j = -\sum_{i=1}^{N} \left[ \frac{d}{dt} \left( \frac{\partial T_i}{\partial \dot{q}_j} \right) - \frac{\partial T_i}{\partial q_j} \right]
\]
Lagrange’s Equations for a Holonomic System

Kane’s equations for a holonomic system with $n$ speeds $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$

\[ Q_j + Q_j^* = 0, \quad j = 1, 2, \ldots, n \]
\[ Q_j^* = -\left[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] \]

Lagrange equations for a holonomic system with $n$ speeds $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j = 0, \quad j = 1, 2, \ldots, n \]

Fundamental form of Lagrange’s equations of motion

Advantages

- Principle of virtual work (Ignore constraint forces)
- One equation of motion for each degree of freedom
- Only need to calculate the kinetic energy (function of velocity)
- Lends itself to automated symbolic manipulation (only scalar quantities)
Fundamental Form of Lagrange’s Equations

Speeds \( \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n \)

\[ n \text{ equations} \quad \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j = 0, \quad j = 1, 2, \ldots, n \]

Matrix equation \( Q + Q^* = 0 \)

\[
\begin{bmatrix}
Q_1 \\
Q_2 \\
\vdots \\
Q_n
\end{bmatrix}
+ \begin{bmatrix}
- \left( \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_1} \right) - \frac{\partial T}{\partial q_1} \right) \\
- \left( \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_2} \right) - \frac{\partial T}{\partial q_2} \right) \\
\vdots \\
- \left( \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_n} \right) - \frac{\partial T}{\partial q_n} \right)
\end{bmatrix} = 0
\]

Only works for speeds \( \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n \)
If we further assume that the forces acting on the system are conservative, we can find a potential function, 

\[ V(q_1, q_2, \ldots, q_n, t) \]

such that all generalized active forces can be expressed as partial derivatives of the potential function:

\[ Q_j = -\frac{\partial V}{\partial q_j}, \quad j = 1, 2, \ldots, n \]

Define the Lagrangian

\[ L = T - V \]

We obtain the Standard Form of Lagrange’s Equations

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0, \quad j = 1, 2, \ldots, n
\]

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j = 0, \quad j = 1, 2, \ldots, n
\]
Example 1: Pendulum on a cart

Coordinates
- $x$
- $\theta$

Speeds
- $\frac{dx}{dt}$
- $\frac{d\theta}{dt}$

Kinetic Energy
$$\frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \left( \dot{x}^2 + l^2 \dot{\theta}^2 + 2l \dot{x} \dot{\theta} \cos \theta \right)$$

Potential Energy
$$-mgl \cos \theta + \frac{1}{2} Kx^2$$
Example 1b: Rigid rod pinned to a cart

Coordinates
- $x$
- $\theta$

Speeds
- $dx/dt$
- $d\theta/dt$

Kinetic Energy

Potential Energy
Maple Code

Standard form of Lagrange's Equations

> Lagrange_Standard := proc(Lagrangian,numvars)
> local k,load,l,lold,p,dt,i; global eq;
> with(linalg):
> l:=Lagrangian;
> lold:=l;
> for i from 1 to numvars do
> l:=subs(diff(q[i](t),t)=utemp,l);
> p[i]:= diff(l,utemp):
> p[i]:=subs(utemp=diff(q[i](t),t),p[i]);
> l:=subs(utemp=diff(q[i](t),t),l);
> l:=subs(q[i](t)=qtemp,l);
> dt[i]:= diff(l,qtemp):
> dt[i]:=subs(qtemp=q[i](t),dt[i]);
> l:=lold;
> od;
> for k from 1 to numvars do
> eq[k]:=diff(p[k],t)-dt[k]=0;
> od;
> end:
Maple Code (Example 1b)

```maple
restart; with(linalg):
> rewriteTime:={seq(q[i](t)=q[i], i=1..2),
   seq(diff(q[i](t),t)=u[i], i=1..2),
   seq(diff(q[i](t),t,t)=udot[i],i=1..2)};
> x:=q[1](t)+b/2*sin(q[2](t));
> y:=b/2*(1-cos(q[2](t)));
> xCart:=q[1](t);
> IG:=m*b*b/12:
> T:=0.5*M*diff(xCart,t)^2 +
   0.5*m*(diff(x, t)^2+diff(y, t)^2) +
   1/2*IG*diff(q[2](t),t)^2:
> V:=m*g*y + 1/2*k*xCart^2;
> L:=T-V:
> Lagrange_Standard(L, 2):
> subs(rewriteTime, simplify(combine(eq[1],trig)));
> subs(rewriteTime, simplify(combine(eq[2],trig)));
```
Example 2: Spherical Pendulum

Generalized coordinates
- $\theta$
- $\phi$

Generalized speeds
- $d\theta/dt$
- $d\phi/dt$

Kinetic Energy

Potential Energy

$z := l \cos(q_1(t)) + l$
A smooth tube containing masses $m_1$ and $m_2$ connected by springs is mounted on a rotating table at an angle $a$. A vertical plane passing through the axis of the tube also passes through the axis of rotation of the table. The table is mounted on an elevator which moves up with an acceleration $a$. Find the equations of motion of the system.