Configuration Space and Degrees of Freedom
Degrees of freedom and constraints

Consider a system $S$ with $N$ particles, $P_r \ (r=1,...,N)$, and their positions vector $x_r$ in some reference frame $A$. The $3N$ components specify the configuration of the system, $S$.

The configuration space is defined as:

$$\mathbb{X} = \left\{ \mathbf{X} \mid \mathbf{X} \in \mathbb{R}^{3N}, \quad \mathbf{X} = [x_1 a_1, x_1 a_2, x_1 a_3, ..., x_N a_1, x_N a_2, x_N a_3]^T \right\}$$

The $3N$ scalar numbers are called configuration space variables or coordinates for the system.

The trajectories of the system in the configuration space are always continuous.
What are Constraints?

Claim (Rosenberg)
In a system of two or more particles, unconstrained motion is simply not possible.

Definition
If the motion of the system is affected by one or more constraints on the positions of the particles, the constraints are called \textit{configuration constraints}. 
A System of Two Particles on a Line

forgotten line

system trajectory

\( x_1 \)

\( x_2 \)
Examples of Configuration Constraints

1. A particle moving on a plane in three dimensional space. The configuration space is $\mathbb{R}^3$. However, the particle is constrained to lie on a plane:
   \[ A x_1 + B x_2 + C x_3 + D = 0 \]

2. A particle suspended from a string in three dimensional space. The configuration space is $\mathbb{R}^3$. The particle is constrained to move so that its distance from a fixed point is always the same. This is called a spherical pendulum.
   \[ (x_1 - a)^2 + (x_2 - b)^2 + (x_3 - c)^2 - r^2 = 0 \]

3. Two particles attached by a massless rod. The configuration space is $\mathbb{R}^6$. The two particles are constrained so that the distance between them is a constant.
   \[ (x_1 - x_4)^2 + (x_2 - x_5)^2 + (x_3 - x_6)^2 - r^2 = 0 \]

4. A particle constrained by the equation below is constrained to move on a circle in three-dimensional space whose radius changes with time $t$.
   \[ x_1 \ dx_1 + x_2 \ dx_2 + x_3 \ dx_3 - c^2 \ dt = 0 \]
Examples of Constraints

Example 1

A particle moving in a horizontal plane (call it the x-y plane) is steered in such a way that the slope of the trajectory is proportional to the time elapsed from $t=0$.

Example 2: Disk rolling on plane

Rolling constraint at $P$

Are these configuration constraints?
Consider a system of $N$ particles, $S$, in a reference frame $A$.

- A $3N$-dimensional configuration space is associated with the system $S$.
- When there are one or more configuration constraints not all of the $3N$ variables describing the system configuration are independent.

The minimum number of variables (also called coordinates) required to completely specify the configuration (position of every particle) of a system is called the number of degrees of freedom for that system.

1. No. of degrees of freedom $= \text{No. of variables required to describe the system}$
   - $\text{No. of independent configuration constraints}$

2. No. of degrees of freedom of a system depends on reference frame.  
   *Controversial!*
Definitions

- **Degrees of freedom of a system**
  The number of independent variables (or coordinates) required to completely specify the configuration of the system.
  - ▼ Point on a plane (2-D)
  - ▼ Point in 3-D space
  - ▼ Line on a plane

- **Kinematic chain**
  A system of rigid bodies connected together by joints. A chain is called closed if it forms a closed loop. A chain that is not closed is called an open chain.
Degrees of Freedom: Example

No. of degrees of freedom = No. of variables required to describe the system - No. of independent configuration constraints
The Planar 3-R manipulator

- Planar kinematic chain
- All joints are revolute with connectivity = 1
- What is the number of degrees of freedom?
Open Kinematic Chains

The Adept 1850 Palletizer

Planar manipulator

No. of degrees of freedom (dof) = no. of independent 1 dof joints.
Connectivity, Mobility, and Degrees of Freedom

- Mobility or number of degrees of freedom of a (general) planar kinematic chain

General expression

\[ M = d(n - j - 1) + \sum_{i=1}^{j} f_i \]

- \( n \) number of links
- \( j \) number of joints between the links
- \( f_i \) connectivity of joint \( i \) (1 for revolute and prismatic, 3 for spherical)
Examples: Mobility (Degrees of Freedom)

Ingersoll Rand machine tool (Stewart Platform)

- number of links, \( n = 20 \)
- number of joints, \( j = 24 \)
- connectivity, \( f_i = 1 \) or \( 3 \)

\[
M = 6(n - j - 1) + \sum_{i=1}^{g} f_i
\]

\[
M = 6(19 - 24) + 36 = 6
\]
Rigidly Connected System of Particles in 3-D

No. of degrees of freedom \( = \) No. of variables required to describe the system
- No. of independent configuration constraints

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<th>( 3N - \binom{N}{2} )</th>
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