Holonomic and Nonholonomic Constraints

Degrees of freedom and constraints

Consider a system S with N particles, P_r (r=1,...,N), and their positions vector $\mathbf{x}_r$ in some reference frame A. The 3N components specify the configuration of the system, S.

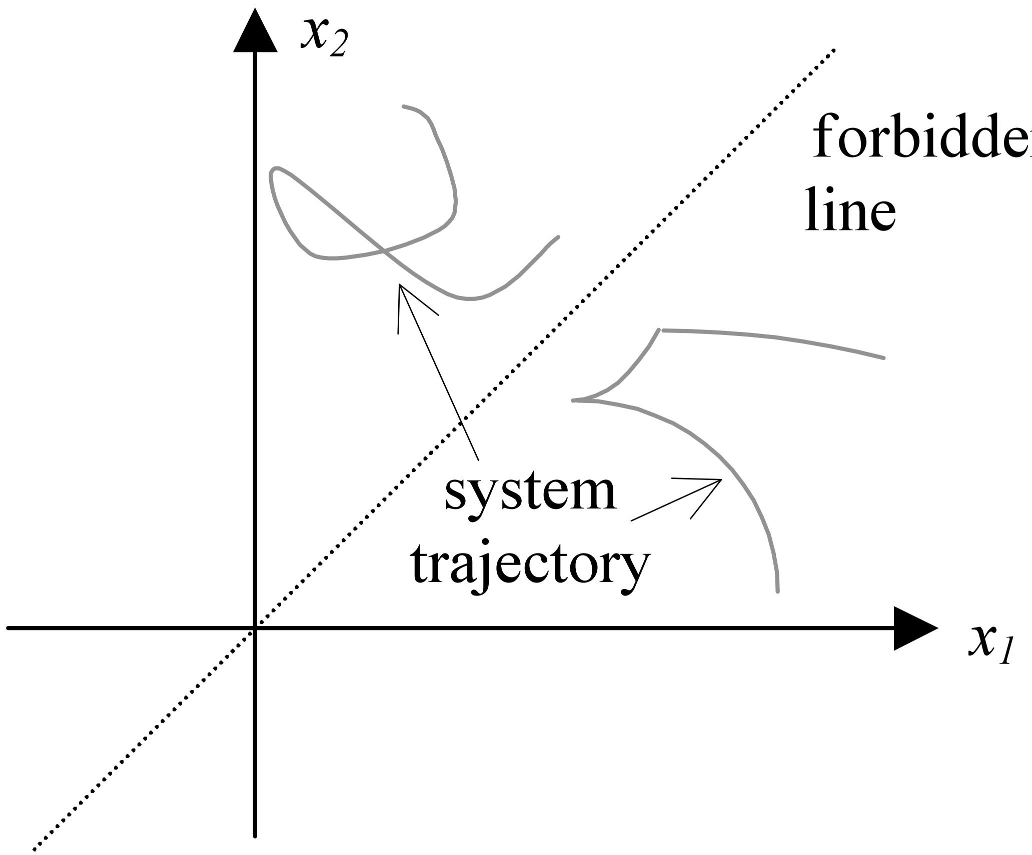
The configuration space is defined as:

$$\aleph = \left\{ \mathbf{X} \mid \mathbf{X} \in R^{3N}, \quad \mathbf{X} = \left[\mathbf{x}_1 \mathbf{a}_1, \mathbf{x}_1 \mathbf{a}_2, \mathbf{x}_1 \mathbf{a}_3, \ldots \mathbf{x}_N \mathbf{a}_1, \mathbf{x}_N \mathbf{a}_2, \mathbf{x}_N \mathbf{a}_3 \right]^T \right\}$$

The 3N scalar numbers are called configuration space variables or coordinates for the system.

The trajectories of the system in the configuration space are always continuous.

A System of Two Particles on a Line



Holonomic Constraints

Constraints on the position (configuration) of a system of particles are called *holonomic* constraints.

- Constraints in which time explicitly enters into the constraint equation are called *rheonomic*.
- Constraints in which time is not explicitly present are called *scleronomic*.

- Particle is constrained to lie on a plane:

  $$A\,x_1 + B\,x_2 + C\,x_3 + D = 0$$

- A particle suspended from a taut string in three dimensional space.

  $$(x_1 - a)^2 + (x_2 - b)^2 + (x_3 - c)^2 - r^2 = 0$$

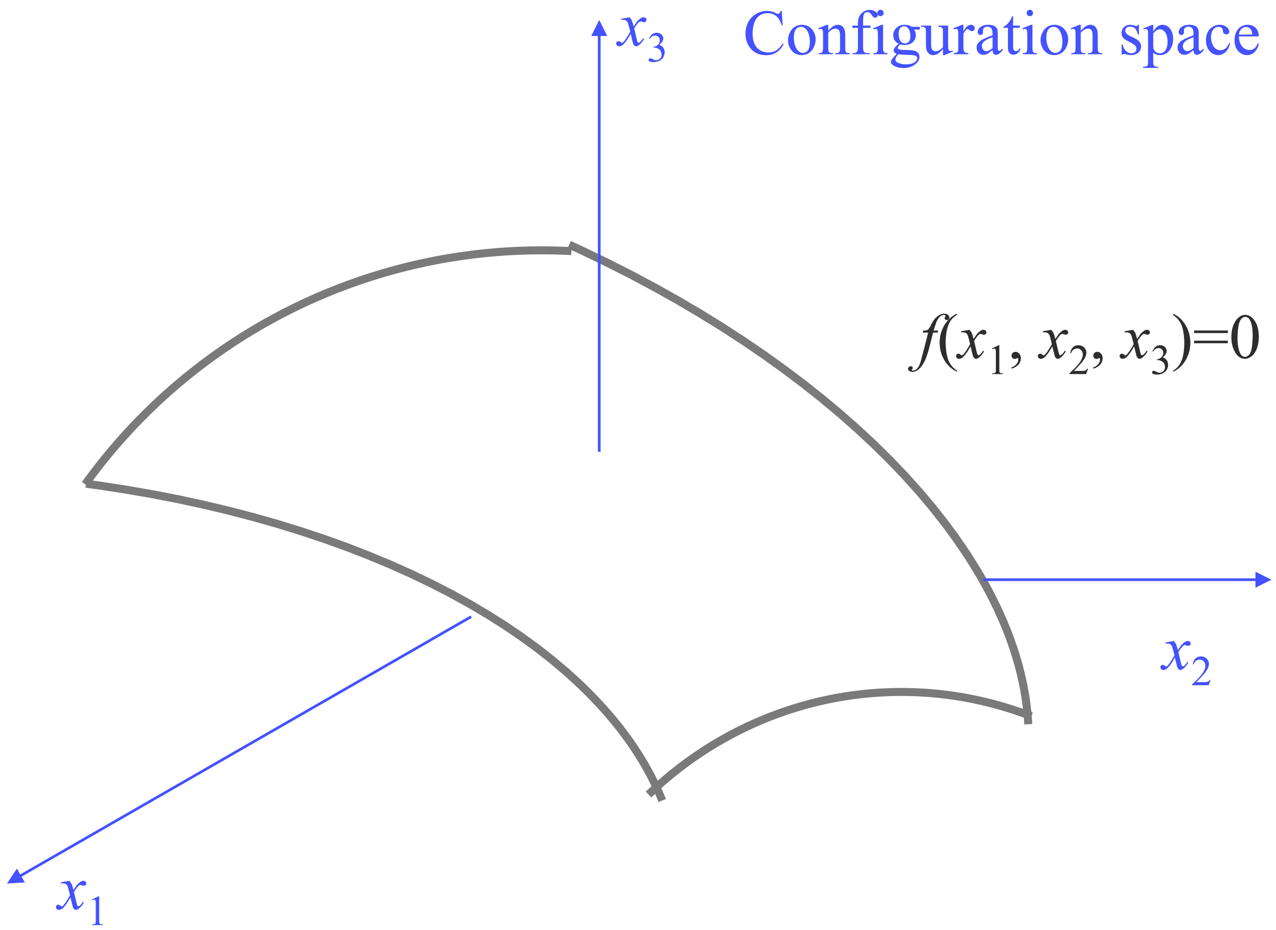
- A particle on spinning platter (carousel)

  $$x_1 = a\cos(\omega t + \phi);\ x_2 = a\sin(\omega t + \phi)$$

- A particle constrained to move on a sphere in three-dimensional space whose radius changes with time t.

  $$x_1\, dx_1 + x_2\, dx_2 + x_3\, dx_3 - c^2\, dt = 0$$

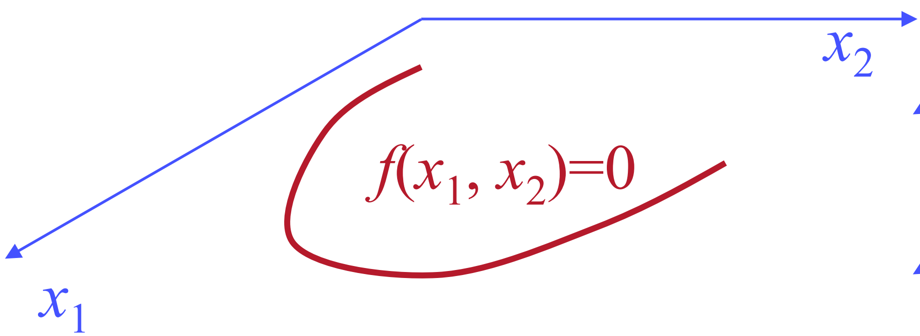
Holonomic Constraint



Scleronomic, holonomic

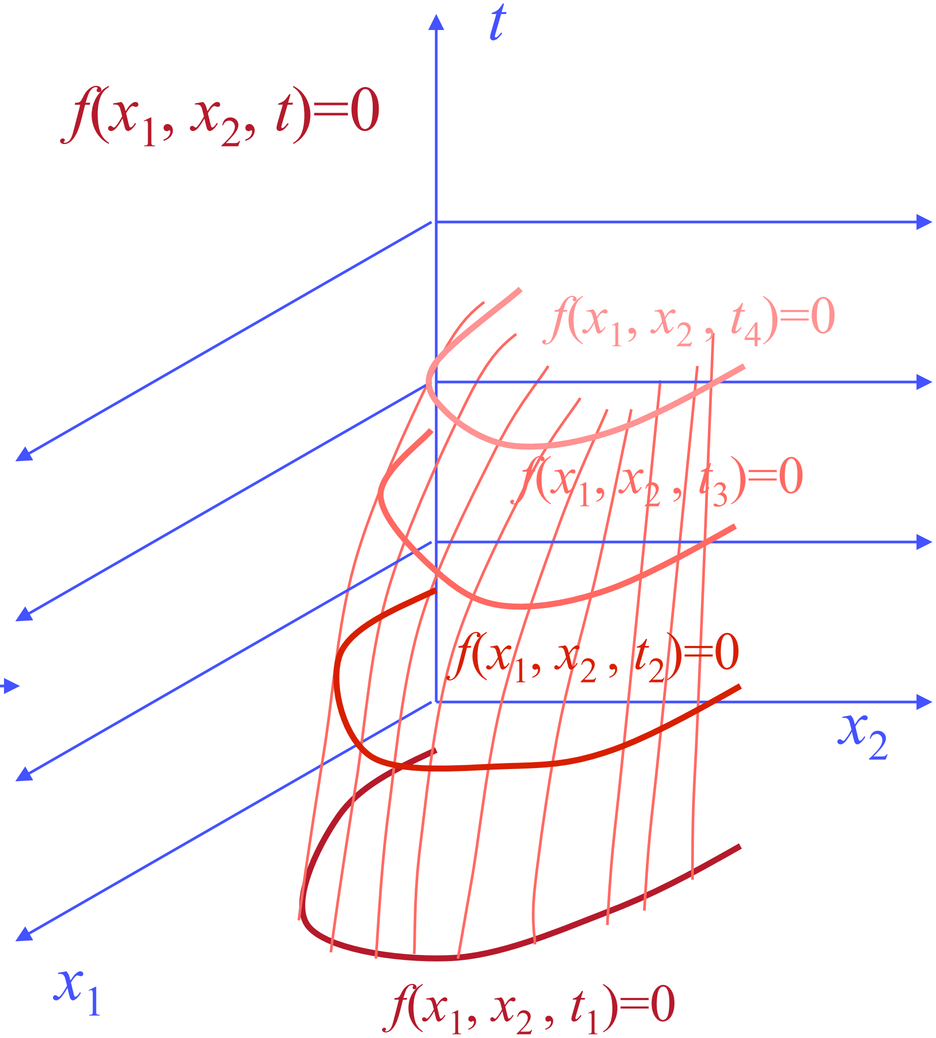
$f(x_1, x_2)=0$

Configuration space



Rheonomic, holonomic

$f(x_1, x_2, t)=0$



Nonholonomic Constraints

Definition 1

- All constraints that are not holonomic

X

- A particle constrained to move on a circle in three-dimensional space whose radius changes with time *t*.

$$x_1 \ dx_1 + x_2 \ dx_2 + \ x_3 \ dx_3 \ - c^2 \ dt = 0$$

Definition 2

- Constraints that constrain the velocities of particles but not their positions

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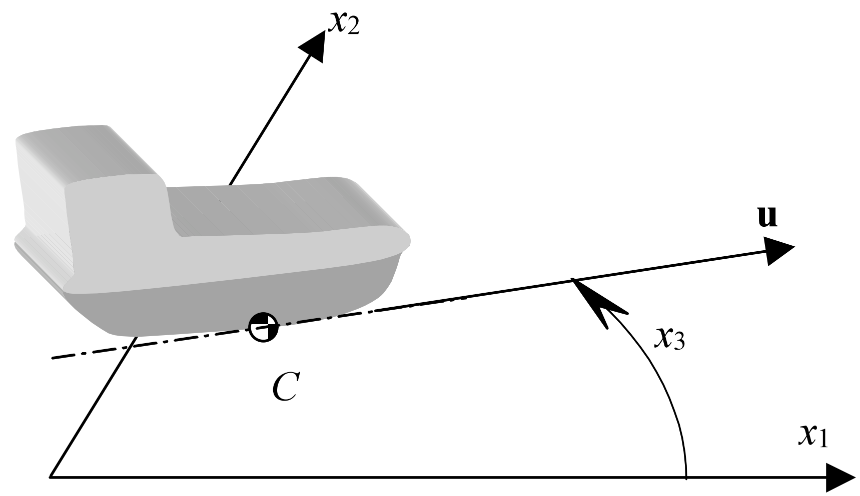
- The *knife-edge constraint*

$$\dot{x}_1 \sin x_3 - \dot{x}_2 \cos x_3 = 0$$

We will use the second definition.



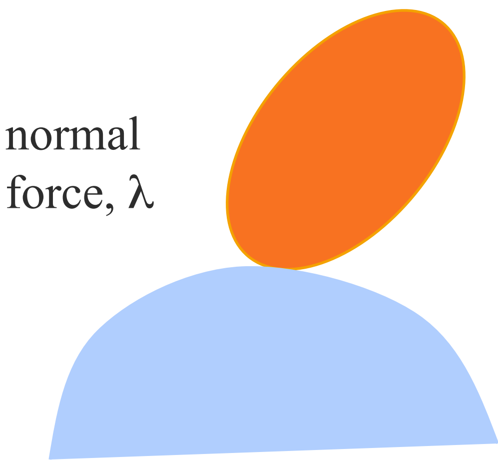
Aside: Inequality Constraints

Holonomic or non holonomic?

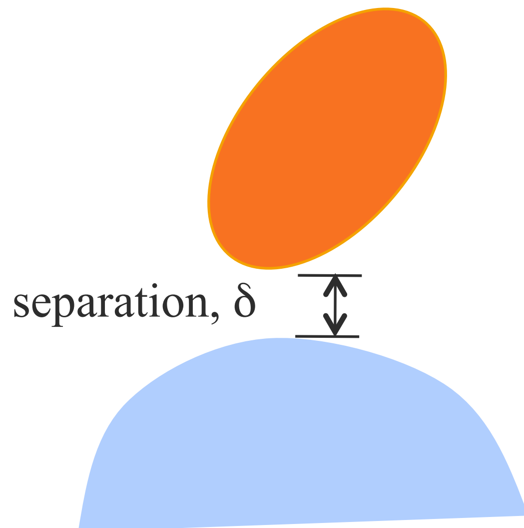
- Inequalities do not constrain the position in the same way as equality constraints do.
- Rosenberg classifies inequalities as nonholonomic constraints.
- We will classify equality constraints into holonomic equality constraints and non holonomic equality constraints and treat inequality constraints separately

Inequalities in mechanics lead to complementarity constraints!

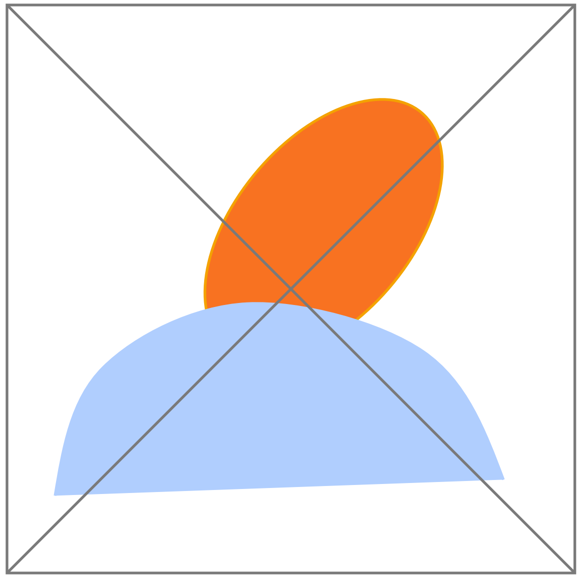
Complementarity Constraints



$\delta = 0, \;\; \lambda > 0$



$\delta > 0, \;\; \lambda = 0$



$\delta < 0$

More generally, $0 \leq \; \delta \perp \lambda \geq \; 0$

Examples of Velocity Constraints

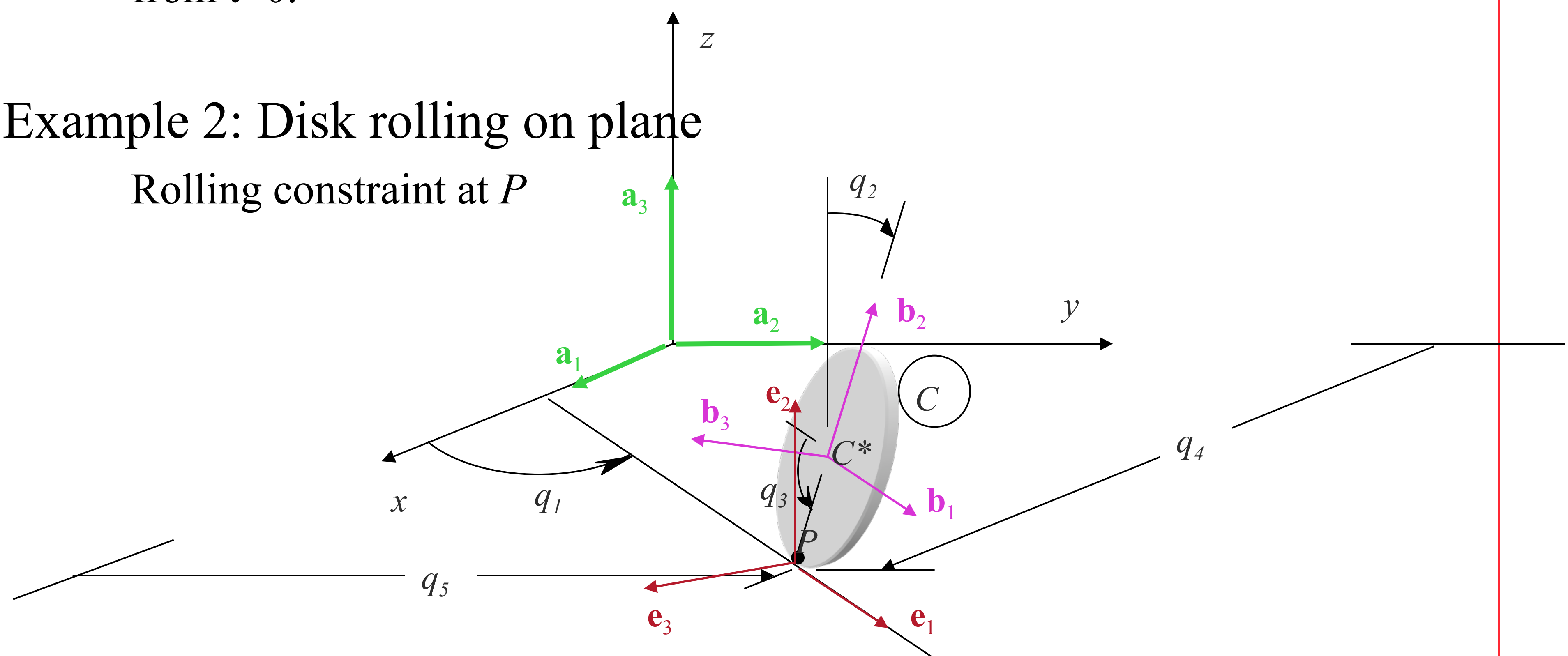
Example 1

Are these configuration constraints?

A particle moving in a horizontal plane (call it the x-y plane) is steered in such a way that the slope of the trajectory is proportional to the time elapsed from t=0.

Example 2: Disk rolling on plane

Rolling constraint at *P*



When is a constraint on the motion nonholonomic?

Velocity constraint

$$a_1\dot{x}_1 + a_2\dot{x}_2 + \ldots + a_{n-1}\dot{x}_{n-1} + a_n = 0$$

Or constraint on instantaneous motion

$$a_1 dx_1 + a_2 dx_2 + \ldots + a_{n-1} dx_{n-1} + a_n dt = 0$$

Pfaffian Form

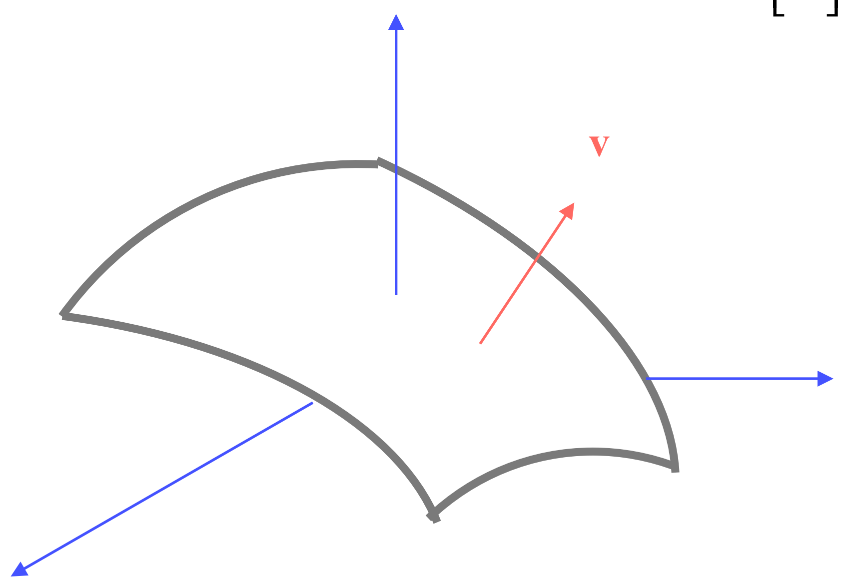
Question

Can the above equation can be reduced to the form:

$$f(x_1, x_2, \ldots, x_{n\text{-}1}, t) = 0$$

We should be familiar with this question in the 3 dimensional case

$$P\,dx + Q\,dy + R dz = 0 \qquad \mathbf{v} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$



Can we construct a surface in 3-D whose normal at every point is given by **v**?

When is a scleronomic constraint on motion in a three-dimensional configuration space nonholonomic?

Velocity constraint

$$P\dot{x} + Q\dot{y} + R\dot{z} = 0$$

Or constraint in the Pfaffian form

$$P\,dx + Q\,dy + R dz = 0 \qquad (1)$$

Question

Can the above equation can be reduced to the form:

$$f(x, y, z) = 0$$

Or,

Can we at least say when the differential form (1) an exact differential?

$$df = P\,dx + Q\,dy + R dz$$

- A **sufficient** condition for (1) to be integrable is that the differential form is an exact differential.
- If it is an exact differential, there must exist a function *f*, such that

$$P = \frac{\partial f}{\partial x}, \qquad Q = \frac{\partial f}{\partial y}, \qquad R = \frac{\partial f}{\partial z}$$

- The necessary and sufficient conditions for this to be true is that the first partial derivatives of *P*, *Q*, and *R* with respect to *x*, *y*, and *z* exist, and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \qquad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}.$$

Recall result from Stokes Theorem!

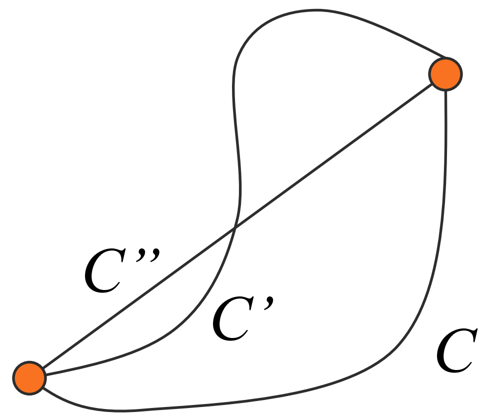
Exactness and independent of path

If **v** is continuous and has continuous first partials in a domain *D, and* the line integral

$$\int_C \mathbf{v} \cdot d\mathbf{r}$$

is independent of path *C* in *D* (that is, **v.**d**r** is *exact*) then

$$\nabla \times \mathbf{v} = 0 \qquad (2)$$



But (2) is only a **sufficient** condition for (1) to be integrable.

Necessary and sufficient condition for a motion constraint in 3-D space to be holonomic

Can the constraint in the Pfaffian form

$$P\,dx + Q\,dy + R dz = 0 \qquad (1)$$

be reduced to the form:

$$f(x, y, z) = 0$$

For the constraint to be *integrable*, it is necessary and sufficient that there exist an integrating factor $\alpha(x, y, z)$, such that,

$$\alpha P\,dx + \alpha Q\,dy + \alpha R dz = 0 \; (3)$$

be an exact differential.

- If (3) is an exact differential, there must exist a function g, such that

$$\alpha P = \frac{\partial g}{\partial x}, \qquad \alpha Q = \frac{\partial g}{\partial y}, \qquad \alpha R = \frac{\partial g}{\partial z}$$

- The necessary and sufficient conditions for this to be true is that the first partial derivatives of P, Q, and R with respect to x, y, and z exist, and

$$\frac{\partial(\alpha P)}{\partial y} = \frac{\partial(\alpha Q)}{\partial x},$$
$$\frac{\partial(\alpha P)}{\partial z} = \frac{\partial(\alpha R)}{\partial x},$$
$$\frac{\partial(\alpha R)}{\partial y} = \frac{\partial(\alpha Q)}{\partial z}.$$

Does there exist α such that:

$$\frac{\partial(\alpha P)}{\partial y} = \frac{\partial(\alpha Q)}{\partial x},$$
$$\frac{\partial(\alpha P)}{\partial z} = \frac{\partial(\alpha R)}{\partial x},$$
$$\frac{\partial(\alpha R)}{\partial y} = \frac{\partial(\alpha Q)}{\partial z}.$$

$$\left(\frac{\partial\alpha}{\partial y}\right)P - \left(\frac{\partial\alpha}{\partial x}\right)Q = \alpha\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right),$$
$$\left(\frac{\partial\alpha}{\partial z}\right)P - \left(\frac{\partial\alpha}{\partial x}\right)R = \alpha\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right),$$
$$\left(\frac{\partial\alpha}{\partial y}\right)R - \left(\frac{\partial\alpha}{\partial z}\right)Q = \alpha\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right).$$

$$\mathbf{v} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

$$\nabla\alpha \times \mathbf{v} = -\alpha\nabla \times \mathbf{v}$$

$$\mathbf{v} \cdot \nabla \times \mathbf{v} = 0$$

Necessary and sufficient condition for (2) to be holonomic, provided **v** is a well-behaved vector field and

Examples

1. $\sin x_3 \, dx_1 - \cos x_3 \, dx_2 = 0$

$$\mathbf{v} = \begin{bmatrix} -\sin x_3 \\ \cos x_3 \\ 0 \end{bmatrix}$$

2. $2x_2x_3 \, dx_1 + x_1x_3 \, dx_2 + x_1x_2 \, dx_3 = 0$

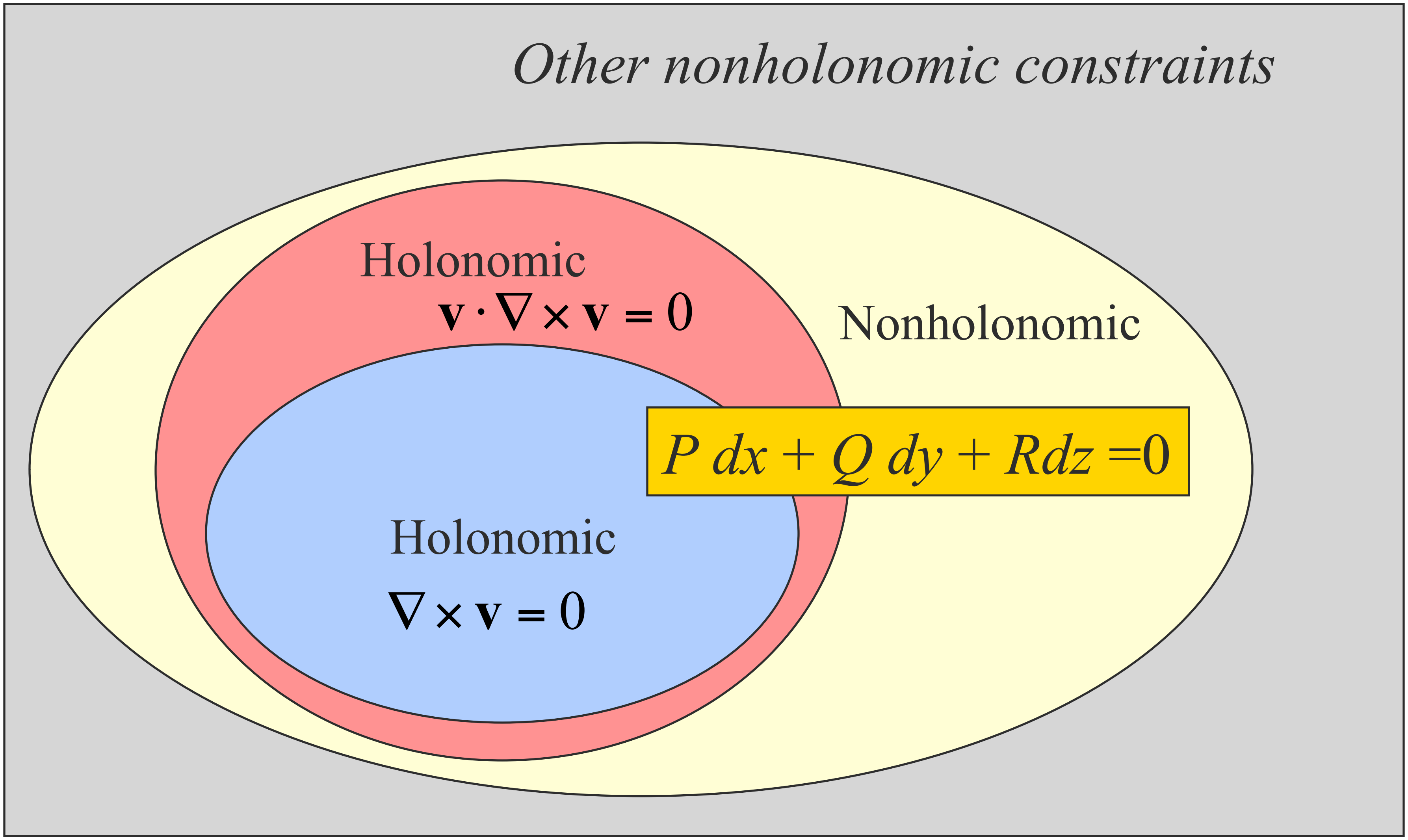
$$x_1 (2x_2x_3 \, dx_1 + x_1x_3 \, dx_2 + x_1x_2 \, dx_3) = 0$$

$$\mathbf{v} = \begin{bmatrix} 2x_2x_3 \\ x_1x_3 \\ x_1x_2 \end{bmatrix}$$

$$d((x_1)^2 x_2 x_3) = 0$$

3. $\dot{x}_1\dot{x}_2 - \dot{x}_3 = 0$

Nonholonomic constraints in 3-D



Extension to 2-D rheonomic constraints

Compare

Can the constraint of the form

$$P\,dx + Q\,dy + R dz = 0$$

be reduced to the form:

$$f(x, y, z) = 0$$

Can the constraint of the form

$$P\,dx + Q\,dy + R dt = 0$$

be reduced to the form:

$$f(x, y, t) = 0$$

Necessary and sufficient condition is the same if you replace z with t

Multiple Constraints

$$dx_2 - x_3\, dx_1 = 0$$

and

$$dx_3 - x_1\, dx_2 = 0$$

Are the constraint equations non holonomic?

Individually: YES!

Together:

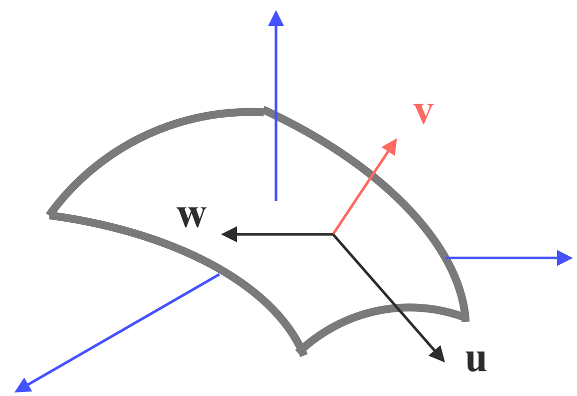
$$dx_3 - x_1\, dx_2 = dx_3 - x_1\,(x_3\, dx_1) = 0 \qquad x_3 = ke^{\frac{x_1^2}{2}}, \qquad x_2 = \int ke^{\frac{x_1^2}{2}} dx_1 + c$$

Frobenius Theorem: Generalization to *n* dimensons

n dimensional configuration space

m independent constraints (i=1,..., m)

$$\sum_{j=1}^{n} a_{ij} dx_j = 0$$



The necessary and sufficient condition for the existence of m independent equations of the form:

$$f_i(x_1, x_2, \ldots, x_n) = 0, \qquad i=1,\ldots, m.$$

is that the following equations be satisfied:

$$\sum_{k=1}^{n}\sum_{l=1}^{n}\left(\frac{\partial a_{il}}{\partial x_k} - \frac{\partial a_{ik}}{\partial x_l}\right) u_k w_l = 0$$

where u_k and w_l are components of any two n vectors that lie in the null space of the mxn coefficient matrix $\mathbf{A} = [a_{ij}]$:

$$\sum_{j=1}^{n} a_{ij} u_j = 0, \quad \sum_{j=1}^{n} a_{ij} w_j = 0,$$

Generalized Coordinates and Number of Degrees of Freedom

Number of degrees of freedom of a holonomic system in any reference frame

- the minimum number of variables to completely specify the position of every particle in the system in the chosen reference

The variables are called generalized coordinates

There can be no holonomic constraint equations that constrain* the values the generalized coordinates can have.

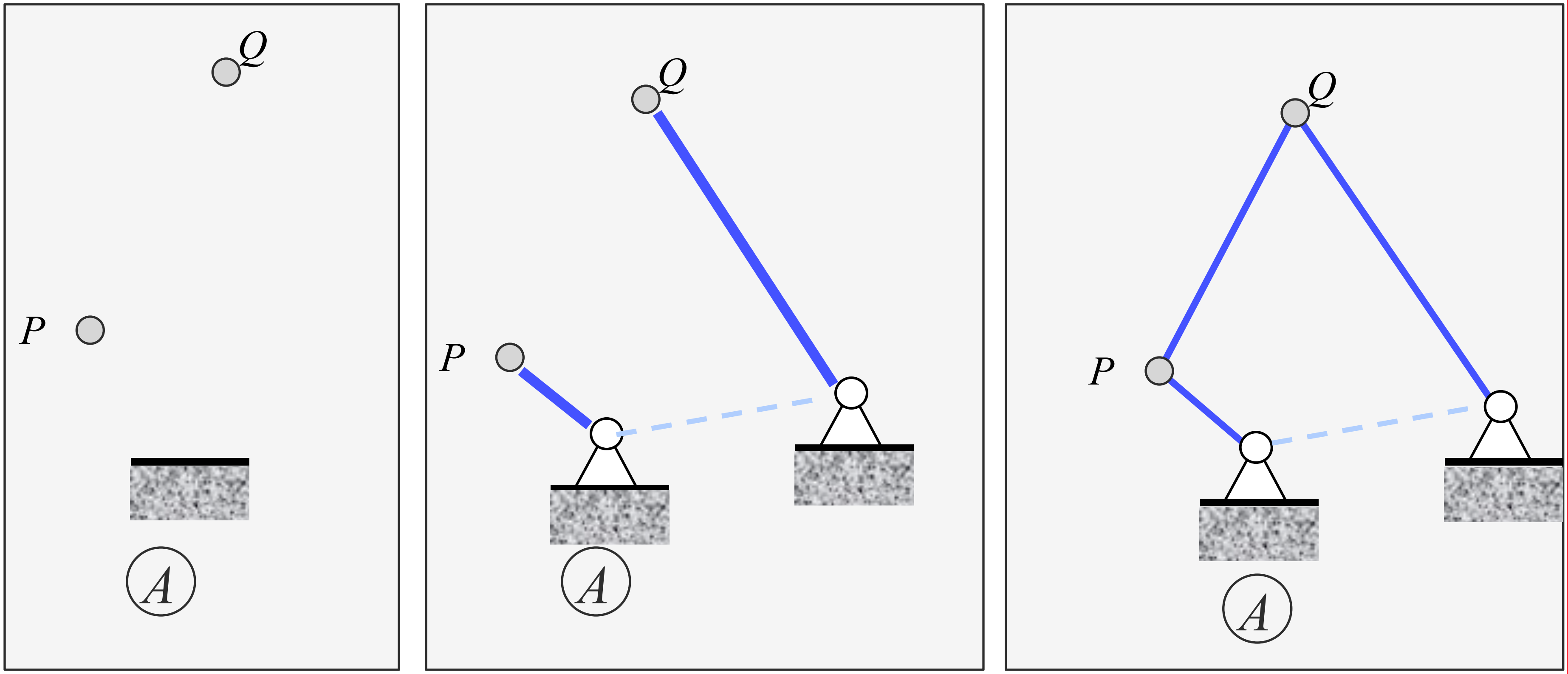
q_1, q_2, ..., q_n denote the generalized coordinates for a system with n degrees of freedom in a reference frame A.

No. of degrees of freedom

= No. of variables required to describe the system

- No. of independent configuration constraints

Degrees of Freedom: Example



No. of degrees of freedom = No. of variables required to describe the system

- No. of independent configuration constraints

Generalized Coordinates and Speeds

Holonomic Systems

Number of degrees of freedom of a system in any reference frame

- the minimum number of variables to completely specify the position of every particle in the system in the chosen reference

The variables are called generalized coordinates

q_1, q_2, ..., q_n denote the generalized coordinates for a system with n degrees of freedom in a reference frame A.

n generalized coordinates specify the position (configuration of the system)

For a holonomic system, the number of independent speeds describing the rate of change of configuration of the system is also equal to n

In a system with n degrees of freedom in a reference frame A, there are n scalar quantities, u_1, u_2, ..., u_n (for that reference frame) called generalized speeds. They that are related to the derivatives of the generalized coordinates by :

$$u_i = \sum_{j=1}^{n} Y_{ij}(q_1, q_2, \ldots, t)\dot{q}_j + Z_i(q_1, q_2, \ldots, t)$$

where the nxn matrix $\mathbf{Y} = [Y_{ij}]$ is non singular and $\mathbf{Z}$ is a nx1 vector.

Example 1

Generalized Coordinates

q_1, q_2, q_3, q_4, q_5

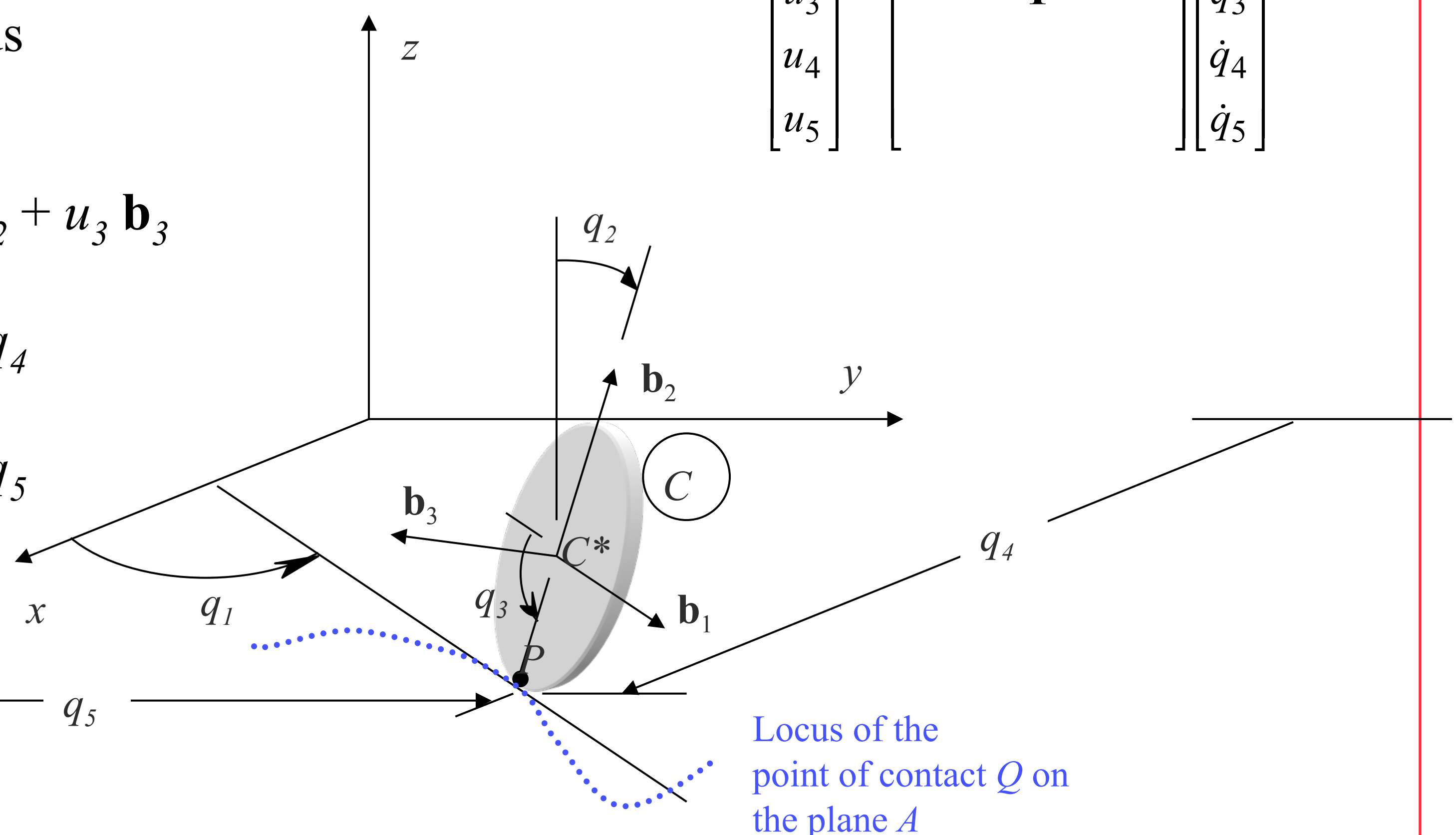
Generalized Speeds

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} & & \\ & \mathbf{Y} & \\ & & \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \end{bmatrix}$$

$^A\boldsymbol{\omega}^C = u_1\,\mathbf{b}_1 + u_2\,\mathbf{b}_2 + u_3\,\mathbf{b}_3$

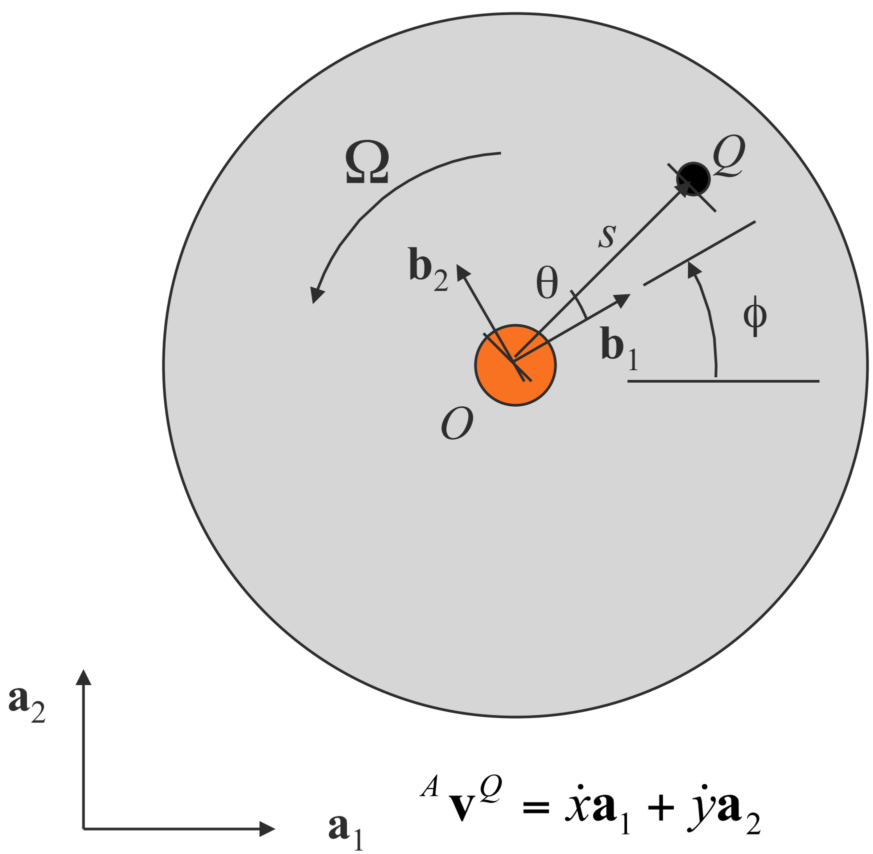
u_4 = derivative of q_4

u_5 = derivative of q_5



Example 2

Bug moving radially on a turntable that can rotate (θ=0)



$${}^{A}\mathbf{v}^{Q} = \dot{x}\mathbf{a}_1 + \dot{y}\mathbf{a}_2$$

Generalized coordinates in A

- s, ϕ
- x_1, x_2

Generalized speeds

$$u_1 = \frac{dx}{dt} \qquad \text{or} \qquad \bar{u}_1 = \frac{dq_1}{dt} = \dot{s}$$

$$u_2 = \frac{dy}{dt} \qquad \qquad \bar{u}_2 = \frac{dq_2}{dt} = \dot{\phi}$$

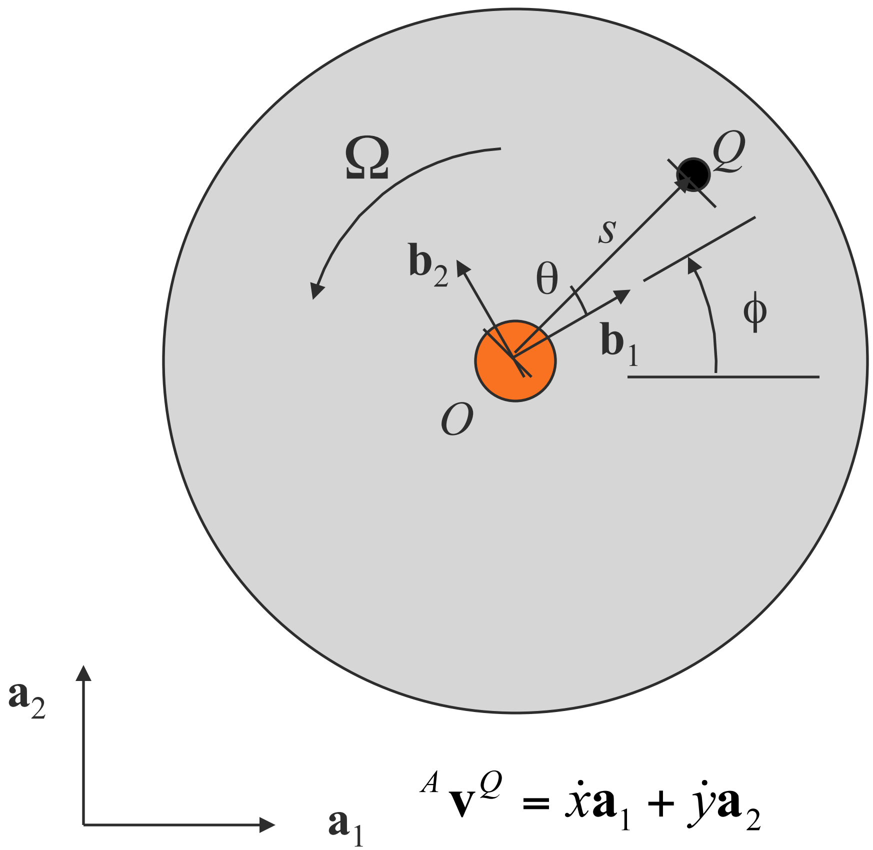
Generalized speeds and derivatives of generalized coordinates

$$u_i = \sum_{j=1}^{n} Y_{ij}(q_1, q_2, \ldots, t)\dot{q}_j + Z_i(q_1, q_2, \ldots, t)$$

Appears in rheonomic constraints

Example 2: Turntable angular velocity is given

Bug moving radially on a rotating turntable (θ=0)



$$^{A}\mathbf{v}^{Q} = \dot{x}\mathbf{a}_1 + \dot{y}\mathbf{a}_2$$

Generalized coordinates in *A*

- s
- x_1

Generalized speeds

$$u_1 = \frac{dx}{dt} \quad \text{or} \quad \bar{u}_1 = \frac{dq_1}{dt} = \dot{s}$$

Generalized speeds and derivatives of generalized coordinates

$$u_i = \sum_{j=1}^{n} Y_{ij}(q_1, q_2, \ldots, t)\dot{q}_j + Z_i(q_1, q_2, \ldots, t)$$

Appears in rheonomic constraints

Nonholonomic Constraints are Written in Terms of Speeds

m constraints in *n* speeds

$$\sum_{j=1}^{n} C_{ij}(q_1, q_2, \ldots, t)u_j + D_i(q_1, q_2, \ldots, t) = 0$$

m speeds are written in terms of the *n*-*m* (*p*) independent speeds

$$u_i = \sum_{k=1}^{p} A_{ik}(q_1, q_2, \ldots, t)u_k + B_i(q_1, q_2, \ldots, t)$$

Define the *number of degrees of freedom for a nonholonomic system* in a reference frame *A* as *p*, the number of independent speeds that are required to completely specify the velocity of any particle belonging to the system, in the reference frame *A*.

Example 3

Number of degrees of freedom

- $n - m = 2$ degrees of freedom

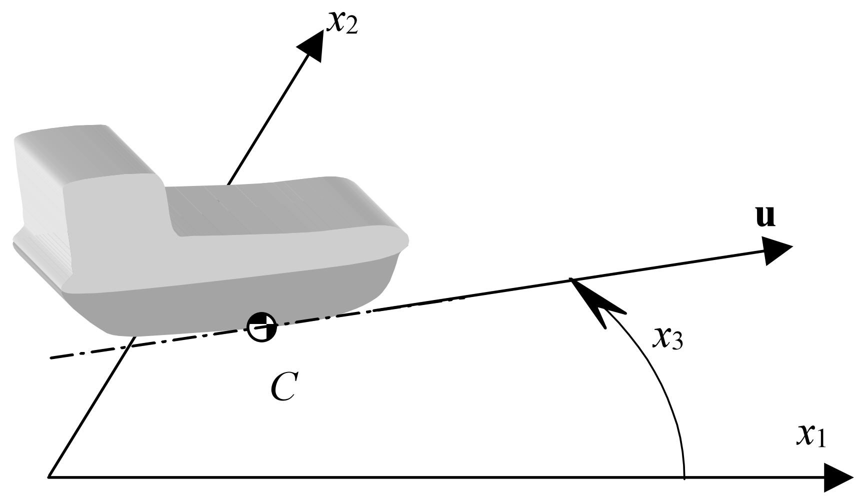
Generalized coordinates

- (x_1, x_2, x_3)

Speeds: Choice 1

- forward velocity along the axis of the skate, v_f
- the speed of rotation about the vertical axis, ω
- and the lateral (skid) velocity in the transverse direction, v_l

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} v_f \\ \omega \\ v_l \end{bmatrix}$$



$$\mathbf{u} = \mathbf{Y}\dot{\mathbf{q}} + \mathbf{Z}$$

$$\mathbf{Y} = \begin{bmatrix} \cos x_3 & \sin x_3 & 0 \\ 0 & 0 & 1 \\ -\sin x_3 & \cos x_3 & 0 \end{bmatrix}, \quad \mathbf{Z} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Speeds: Choice 2 $\quad \mathbf{u} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}$