Optical Flow: 2D point correspondences

\[ I(x + u, y + v, t + 1) = I(x, y, t) \]

(assumption)
Dense optical flow encodes object motion
Dense optical flow encodes object motion
\[ I(x) \]
\[ t = 0 \]

\[ J(x) \]
\[ t = 1 \]
\[ I(x) = J(x + d) \]
\[ I(x) = J(x + d) \]
Correspondence cost

\[
\min_d E = \left\| J(x + d) - I(x) \right\|^2
\]

\[
E(d=(0,0)) = \left\| \begin{array}{c} \circ \end{array} - \begin{array}{c} \circ \end{array} \right\|
\]

\[
E(d=(-7,-9)) = \left\| \begin{array}{c} \circ \end{array} - \begin{array}{c} \circ \end{array} \right\|
\]
Three steps for solving this problem

1. Solve for nonlinear least square solution: \( \frac{dE}{dd}\bigg|_{d=d^*} = 0 \)

2. Taylor expansion on image \( J(x + d) \)

3. Solve for displacement, warp image, and iterate
Step 1: \[ \frac{dE}{dd} \bigg|_{d=d^*} = 0 \]

\[ E(d) = \| J(x + d) - I(x) \|^2 \]

\[ E(d) = (J(x + d) - I(x))^T (J(x + d) - I(x)) \]
Step 1: \[
\frac{dE}{dd}\bigg|_{d=d^*} = 0
\]

\[
E(d) = \left\| J(x + d) - I(x) \right\|^2
\]

\[
E(d) = (J(x + d) - I(x))^T (J(x + d) - I(x))
\]

\[
\frac{\partial E}{\partial d} = 2 \frac{\partial J(x + d)^T}{\partial d} (J(x + d) - I(x))
\]
Step 1: \[ \frac{dE}{dd} \bigg|_{d=d^*} = 0 \]

\[ E(d) = \| J(x+d) - I(x) \|^2 \]

\[ E(d) = (J(x+d) - I(x))^T (J(x+d) - I(x)) \]

\[ \frac{\partial E}{\partial d} = 2 \frac{\partial J(x+d)}{\partial d}^T (J(x+d) - I(x)) \]

\[ \frac{\partial E}{\partial d} = 2 \frac{\partial J(x)}{\partial x}^T (J(x+d) - I(x)) \]
Step 1: \[ \frac{dE}{dd}\bigg|_{d=d^*} = 0 \]

\[ E(d) = \| J(x + d) - I(x) \|^2 \]

\[
\frac{\partial E}{\partial d} = 2 \frac{\partial J(x)}{\partial x} \left( J(x + d) - I(x) \right)
\]

where \[ \frac{\partial J(x)}{\partial x} = \left[ \frac{\partial J(x, y)}{\partial x}, \frac{\partial J(x, y)}{\partial y} \right] \] : Image Gradient
Step 1: \[ \frac{dE}{dd}|_{d=d^*} = 0 \]

\[ E(d) = \left\| J(x + d) - I(x) \right\|^2 \]

\[ \frac{\partial E}{\partial d} = 2 \frac{\partial J(x)^T}{\partial x} (J(x + d) - I(x)) \]

\[ \frac{\partial E}{\partial d} = 2 \frac{\partial J(x)^T}{\partial x} (J(x + d) - I(x)) = 0 \]

Find \( d \) such that the above equation is satisfied
Step 1: \[ \frac{dE}{dd}\bigg|_{d=d^*} = 0 \]

\[ \frac{\partial E}{\partial d} = 2 \frac{\partial J(x)^T}{\partial x} (J(x + d) - I(x)) = 0 \]

Find \( d \) such that the above equation is satisfied.
Nonlinear System

Find $d$ such that the above equation is satisfied

$$\frac{\partial E}{\partial d} = 2 \frac{\partial J(x)^T}{\partial x} (J(x + d) - I(x)) = 0$$

Idea: how to predict an image when it is shifted by $\Delta d$

This is a nonlinear process, easy to carry out by image warping, but not easy to write down as an equation.
Nonlinear System

Find $d$ such that the above equation is satisfied

$$\frac{\partial E}{\partial d} = 2 \frac{\partial J(x)^T}{\partial x} \left( J(x + d) - I(x) \right) = 0$$

Idea: how to predict an image when it is shifted by $\Delta d$

Taylor expansion:

$$J(x + \Delta d) = J(x) + \frac{\partial J(x)}{\partial x} \Delta d + \text{H.O.T.}$$
Step 2: Taylor expansion

\[ J(x + \Delta d) = J(x) + \frac{\partial J(x)}{\partial x} \Delta d + \text{H.O.T.} \]
Step 2: Taylor expansion

Find $d$ such that the above equation is satisfied

$$\frac{\partial E}{\partial d} = 2 \frac{\partial J(x)^T}{\partial x} \left( J(x + d) - I(x) \right) = 0$$

$$J(x + \Delta d) = J(x) + \frac{\partial J(x)}{\partial x} \Delta d + H.O.T.$$

$$\frac{\partial J(x)^T}{\partial x} \left( \frac{\partial J(x)}{\partial x} \Delta d \right) = \frac{\partial J(x)^T}{\partial x} (I(x) - J(x))$$
Step 2: Taylor expansion

Find $d$ such that the above equation is satisfied

$$\frac{\partial E}{\partial d} = 2 \frac{\partial J(x)^T}{\partial x} \left( J(x + d) - I(x) \right) = 0$$

$$J(x + \Delta d) = J(x) + \frac{\partial J(x)}{\partial x} \Delta d + H.O.T.$$
\[
\Delta \mathbf{d} = \left( \frac{\partial J(\mathbf{x})^T}{\partial \mathbf{x}} - \frac{\partial J(\mathbf{x})}{\partial \mathbf{x}} \right) \cdot \frac{\partial J(\mathbf{x})^T}{\partial \mathbf{x}} \left( I(\mathbf{x}) - J(\mathbf{x}) \right)
\]

2D unknowns flow vector per pixel, 2 equations
\[ \Delta d = \frac{\partial J(x)^T}{\partial x} \left( I(x) - J(x) \right) \]

2D unknowns flow vector per pixel, 2 equations

Also known as second moment matrix
\[
\begin{pmatrix}
\frac{\partial J(x)}{\partial x} & \frac{\partial J(x)}{\partial x} \\
\frac{\partial J(x)}{\partial x} & \frac{\partial J(x)}{\partial x}
\end{pmatrix}
\]
\[
\frac{\partial J(x)^T}{\partial x} (I(x) - J(x))
\]
\[
\left( \frac{\partial J(x)^T}{\partial x} \frac{\partial J(x)}{\partial x} \right) \Delta d = \frac{\partial J(x)^T}{\partial x} \left( I(x) - J(x) \right)
\]
\[
\left( \frac{\partial J(x)^T}{\partial x} \frac{\partial J(x)}{\partial x} \right) \Delta d = \frac{\partial J(x)^T}{\partial x} \left( I(x) - J(x) \right)
\]

\[
d_x + d_y = \text{(image)}
\]
Solve for displacement: $d = (-7, -9)$
\[ d_x + d_y = \]

[Diagram showing visual representations of \(d_x\) and \(d_y\) with graphical elements.]
\[
\begin{align*}
\Delta \mathbf{d} &= \frac{\partial J(\mathbf{x})^T}{\partial \mathbf{x}} (l(\mathbf{x}) - J(\mathbf{x})) \\
\mathbf{d}_x \begin{pmatrix}
0 \\
\triangle \\
0 \\
\triangle
\end{pmatrix} + \mathbf{d}_y \begin{pmatrix}
0 \\
\triangle \\
0 \\
\triangle
\end{pmatrix} &= \begin{pmatrix}
0 \\
\triangle \\
0 \\
\triangle
\end{pmatrix}
\end{align*}
\]

Cannot solve for the displacement
\[
\min E = \sum_{d, x \in W} \| J(x + d) - I(x) \|^2
\]

Pooling over a window
\[
\min_d E = \sum_{x \in W} \left\| J(x + d) - I(x) \right\|^2
\]

\[
\sum_{x \in W} \left( \frac{\partial J(x)^T}{\partial x} \frac{\partial J(x)}{\partial x} \right) \Delta d = \sum_{x \in W} \frac{\partial J(x)^T}{\partial x} \left( I(x) - J(x) \right)
\]
Summing over pixels

2 × 2 matrix
Summing over pixels

2 × 1 matrix
\[
\sum_{x \in W} \left( \frac{\partial J(x)^T}{\partial x} \frac{\partial J(x)}{\partial x} \right) \Delta d = \sum_{x \in W} \frac{\partial J(x)^T}{\partial x} (l(x) - J(x))
\]
Step 3: Solve for displacement, warp image, and iterate

\[ I(x) \quad J(x) \quad \text{Error} \]

\[ d = (-7, -9) \]
Step 3: Solve for displacement, warp image, and iterate

\[ \mathbf{d} = (-7, -9) \]

\[ \mathbf{J}^{t=1}(\mathbf{x}) = \mathbf{J}(\mathbf{x} + \mathbf{d}) \]
\[ \Delta d = \nabla J(x)^T \left( I(x) - J(x) \right) \]
\[
\Delta d = \left( \frac{\partial J(x)^T}{\partial x}, \frac{\partial J(x)}{\partial x} \right) \frac{\partial J(x)^T}{\partial x} (I(x) - J(x))
\]

\[
\frac{\delta J(x)}{\delta x} = \begin{array}{c}
\end{array}
\]

\[
\frac{\delta J(x)}{\delta y} = \begin{array}{c}
\end{array}
\]
\[
\left( \frac{\partial J(x)^T}{\partial x} \frac{\partial J(x)}{\partial x} \right) \Delta d = \frac{\partial J(x)^T}{\partial x} \left( I(x) - J(x) \right)
\]
Step 3: Solve for displacement, warp image, and iterate

\[ \mathbf{I}(\mathbf{x}) \quad \rightarrow \quad \mathbf{d}=(-4.9, -0.4) \quad \rightarrow \quad \mathbf{J}^{t=1}(\mathbf{x}) = \mathbf{J}(\mathbf{x} + \mathbf{d}) \quad \rightarrow \quad \mathbf{J}(\mathbf{x}) \quad \rightarrow \quad \text{Error} \]
$d = (-4.9, -0.4)$

$\rightarrow$ $d = (-0.1, -5.8)$

$\leftarrow$ $d = (0, -3.7)$

Error
A Failed Case: fast movement
\[
\Delta d = \left( \frac{\partial J(x)^T}{\partial x} \frac{\partial J(x)}{\partial x} \right) \Delta d = \frac{\partial J(x)^T}{\partial x} \left( I(x) - J(x) \right)
\]

A Failed Case: fast movement

\[I(x) = J(x + d)\]

\[J(x)\]
\[ I(x) = J(x + d) \]
The influence of \( l(x) \) is not incorporated!
The influence of $l(x)$ is not incorporated!
\[
\left( \frac{\partial J(x)^T}{\partial x} \frac{\partial J(x)}{\partial x} \right) \Delta d = \frac{\partial J(x)^T}{\partial x} \left( l(x) - J(x) \right)
\]

The influence of \( l(x) \) is Not included!

Guess what’s the corresponding displacement?
$$\begin{align*}
\Delta \mathbf{d} &= \frac{\partial J(\mathbf{x})^T}{\partial \mathbf{x}} \left( I(\mathbf{x}) - J(\mathbf{x}) \right) \\
\frac{\delta J(x)}{\delta x} + \frac{\delta J(x)}{\delta y} &= \frac{\delta J(x)^2}{\delta x} + \frac{\delta J(x)^2}{\delta y} \\
\frac{\delta J(x)}{\delta x} \frac{\delta J(x)}{\delta y} &= \frac{\partial J(x)}{\partial x} \left( I(x) - J(x) \right)
\end{align*}$$
Summing over pixels

\[ \frac{\partial J(x)^T}{\partial x} \frac{\partial J(x)}{\partial x} \]

2 \times 2 matrix

Summing over pixels

\[ \frac{\partial J(x)^T}{\partial x} (I(x) - J(x)) \]

2 \times 1 matrix
\[
\Delta d = \left( \frac{\partial J(x)^T}{\partial x} \frac{\partial J(x)}{\partial x} \right) \left[ \begin{array}{c} d_x \\ d_y \end{array} \right] = \frac{\partial J(x)^T}{\partial x} \left( I(x) - J(x) \right)
\]

\([-0.011, 0.09]\)

Almost zero motion, why?
The influence of $l(x)$ is not incorporated!
Solution 1: multi scale optical flow

Solution 2: increase the kernel size of gradient operator
Solution 1: multi scale optical flow

\[ I(x) = J(x + d) \]
\[
\left( \frac{\partial J_{\downarrow 4}(x)^T}{\partial x}, \frac{\partial J_{\downarrow 4}(x)}{\partial x} \right) \Delta d = \frac{\partial J_{\downarrow 4}(x)^T}{\partial x} \left( I_{\downarrow 4}(x) - J_{\downarrow 4}(x) \right)
\]

Start from a coarser resolution image
\[
\frac{\partial J_{\downarrow 4}(x)^T}{\partial x} \left( I_{\downarrow 4}(x) - J_{\downarrow 4}(x) \right) \frac{\partial J_{\downarrow 4}(x)}{\partial x} \frac{-J_{\downarrow 4}(x)}{I_{\downarrow 4}(x)}
\]

The influence of \( I(x) \) is incorporated!
\[
\frac{\partial J_{\downarrow 4}(x)^T}{\partial x}\left(I_{\downarrow 4}(x) - J_{\downarrow 4}(x)\right)
\]

The influence of \(I(x)\) is incorporated!
\[
\left( \frac{\partial J_{\downarrow 4}(x)^T}{\partial x} \frac{\partial J_{\downarrow 4}(x)}{\partial x} \right) \Delta d = \frac{\partial J_{\downarrow 4}(x)^T}{\partial x} \left( I_{\downarrow 4}(x) - J_{\downarrow 4}(x) \right)
\]
Summing over pixels

\[
\begin{bmatrix}
\partial J_{\downarrow 4}(x)^T \\
\partial x
\end{bmatrix}
\begin{bmatrix}
\partial J_{\downarrow 4}(x) \\
\partial x
\end{bmatrix}
\]

\[
\frac{\partial J_{\downarrow 4}(x)^T}{\partial x} \left( l_{\downarrow 4}(x) - J_{\downarrow 4}(x) \right)
\]

2 x 2 matrix

Summing over pixels

2 x 1 matrix
\[
\Delta \mathbf{d} = \frac{\partial J_{\downarrow 4}(\mathbf{x})^T}{\partial \mathbf{x}} \left( I_{\downarrow 4}(\mathbf{x}) - J_{\downarrow 4}(\mathbf{x}) \right)
\]

\[d_x + d_y = \begin{bmatrix} -3.3 \\ -3.0 \end{bmatrix}\]
$J(x)$

$\downarrow 4$

$1/4$

$J_{\downarrow 4}(x)$
Down sampling image is equivalent to increase the kernel size.
\[ \mathbf{I}(x) \]
\[ t = 0 \]

\[ \mathbf{J}(x) \]
\[ t = 1 \]
\[
\begin{pmatrix}
\frac{\partial J(x)^T}{\partial x} & \frac{\partial J(x)}{\partial x}
\end{pmatrix}
\Delta d = \frac{\partial J(x)^T}{\partial x} \left( I(x) - J(x) \right)
\]
Solution 2: increase the kernel size of gradient operator
Solution 2: increase the kernel size of gradient operator
Solution 2: increase the kernel size of gradient operator
Solution 2: increase the kernel size of gradient operator
small kernel

\[
\frac{\partial J(x)}{\partial x} (I(x) - J(x)) \quad \times
\]

\[
\left( \frac{\partial J(x)}{\partial x} \right)^T (I(x) - J(x))
\]
Median kernel

\[
\frac{\partial J(\mathbf{x})}{\partial x}^T \left( I(\mathbf{x}) - J(\mathbf{x}) \right)
\]

\[
\frac{\partial J(\mathbf{x})}{\partial x}
\]

\[
I(\mathbf{x}) - J(\mathbf{x})
\]
Large kernel

\[
\frac{\partial J(x)^T}{\partial x} (l(x) - J(x))
\]

\[
\frac{\partial J(x)}{\partial x}
\]

\[
l(x) - J(x)
\]
\[
\left( \frac{\partial J(x)^T}{\partial x}, \frac{\partial J(x)}{\partial x} \right) \Delta d = \frac{\partial J(x)^T}{\partial x} (I(x) - J(x))
\]

\[d = (-0.6, 1.1)\]
\[
\left( \frac{\partial J(x)^T}{\partial x} \frac{\partial J(x)}{\partial x} \right) \Delta d = \frac{\partial J(x)^T}{\partial x} (l(x) - J(x))
\]

\[
d = (\Delta d_x, \Delta d_y) = (-2.9, -3.0)
\]
\[
\begin{pmatrix}
\frac{\partial J(x)^T}{\partial x} & \frac{\partial J(x)}{\partial x}
\end{pmatrix}
\Delta d = \frac{\partial J(x)^T}{\partial x} (l(x) - J(x))
\]

\[d = (-8.3, 19.0)\]
Larger kernel does not mean the better kernel!
Proper kernel size is important!