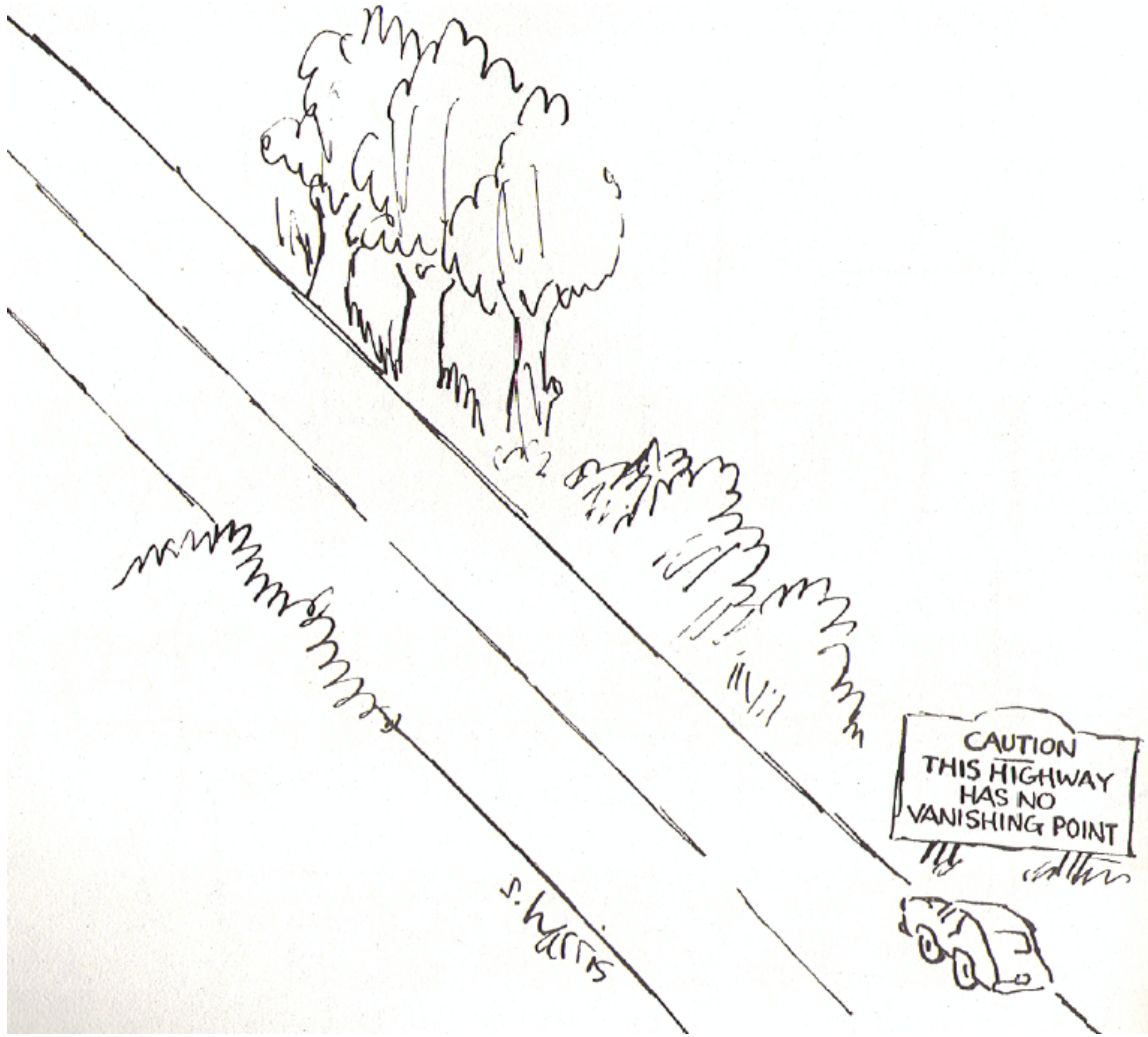
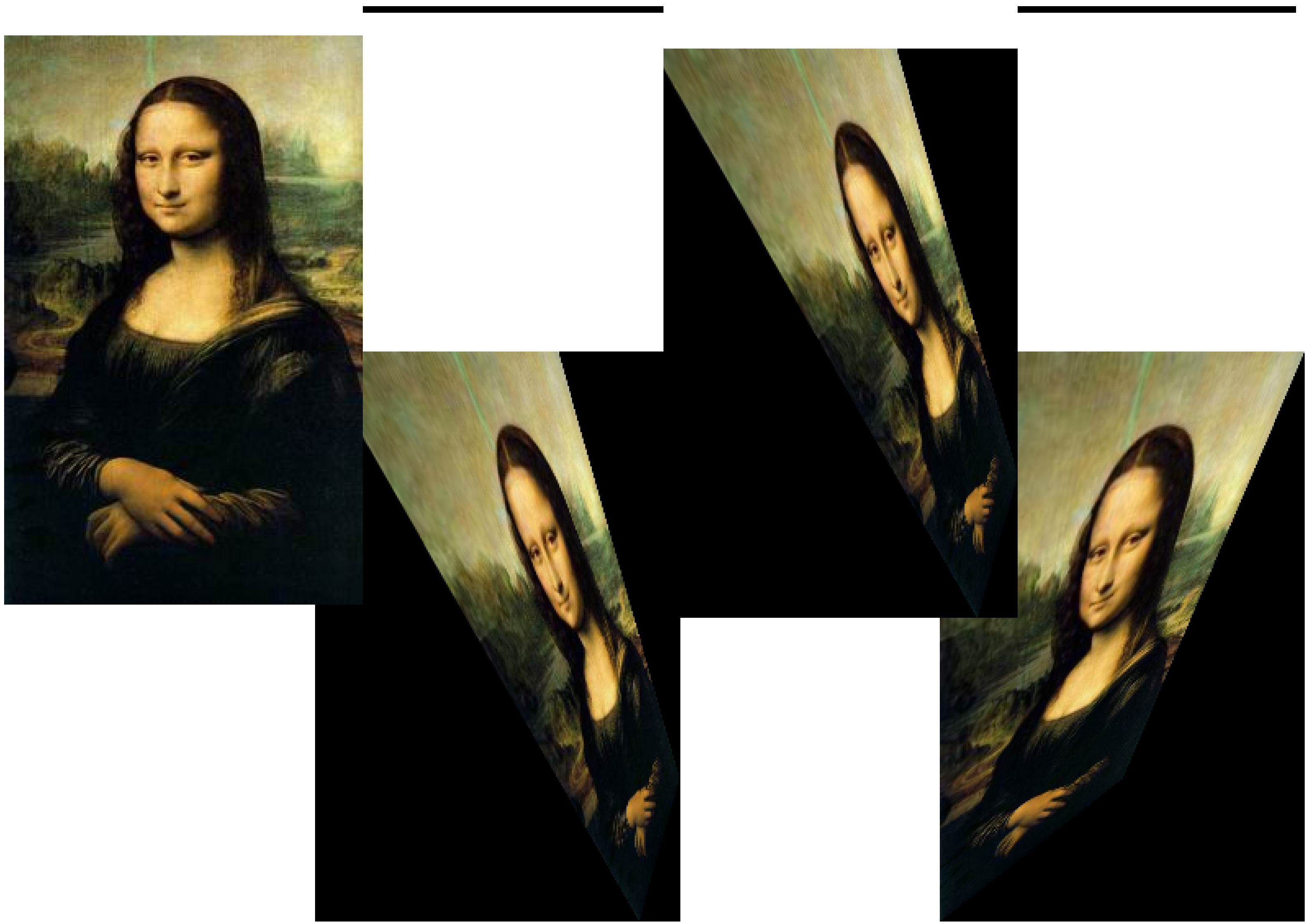
Image Warping,
Linear Algebra

CIS581

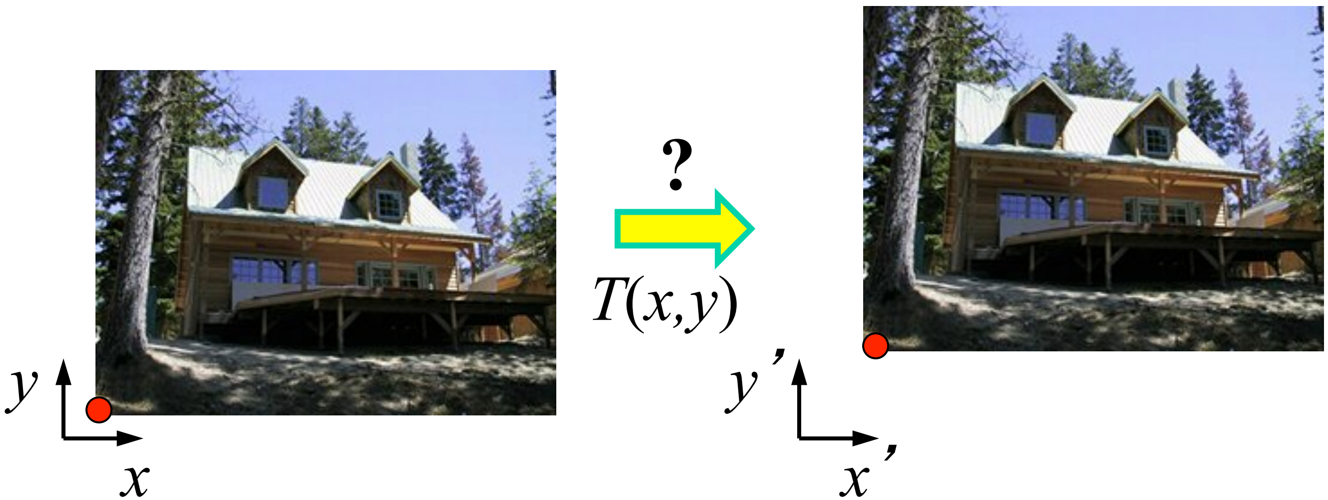


From Plane to Plane



Degree of freedom

Translation: # correspondences?



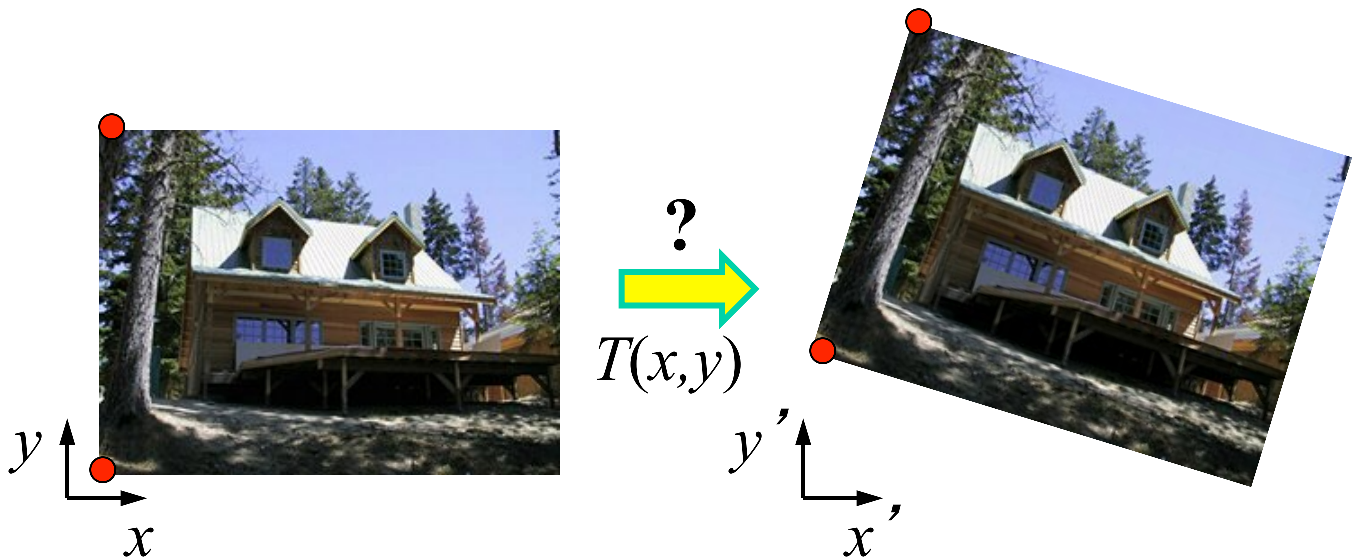
How many correspondences needed for translation?

How many Degrees of Freedom?

What is the transformation matrix?

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & p'_x - p_x \\ 0 & 1 & p'_y - p_y \\ 0 & 0 & 1 \end{bmatrix}$$

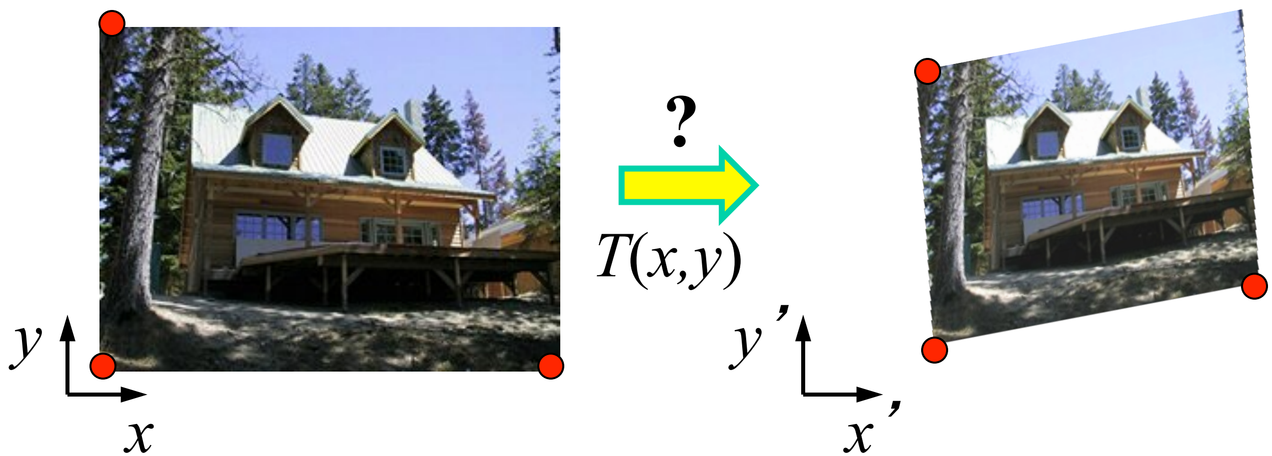
Euclidian: # correspondences?



How many correspondences needed for translation+rotation?

How many DOF?

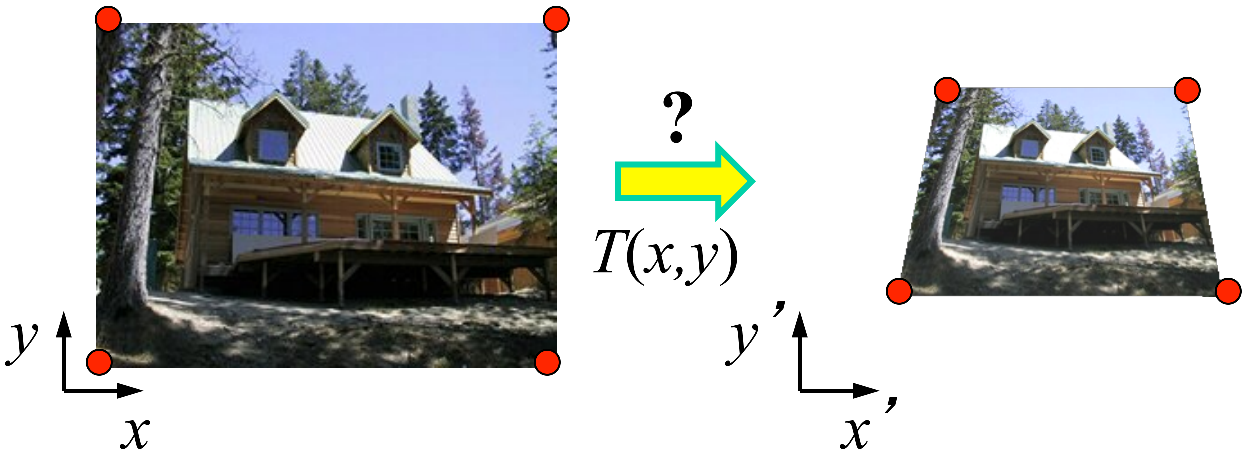
Affine: # correspondences?



How many correspondences needed for affine?

How many DOF?

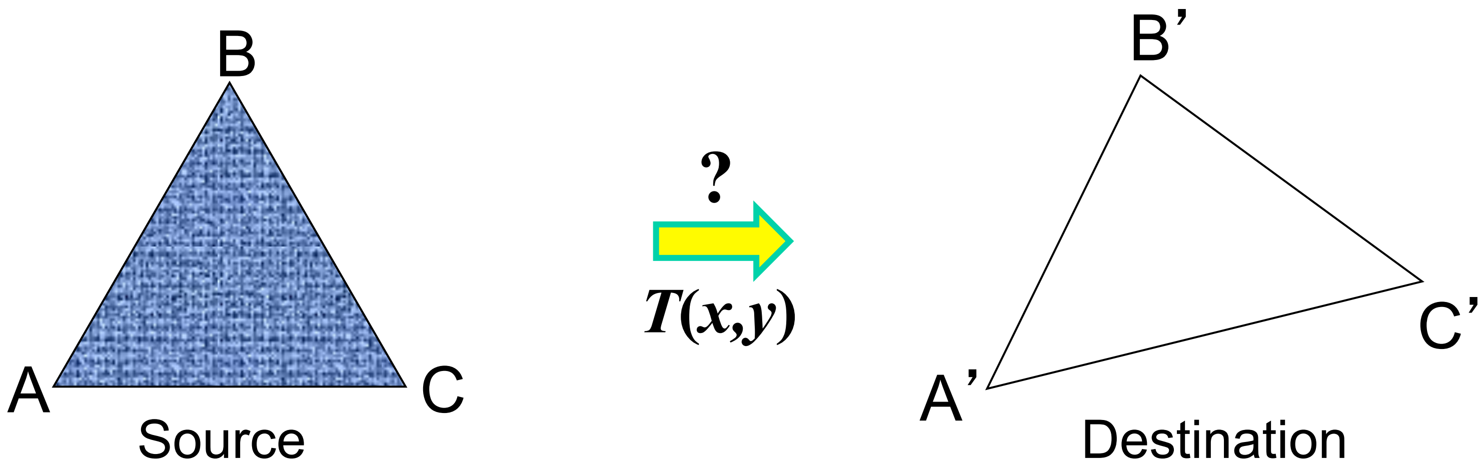
Projective: # correspondences?



How many correspondences needed for projective?

How many DOF?

Example: warping triangles



Given two triangles: ABC and A'B'C' in 2D (12 numbers)

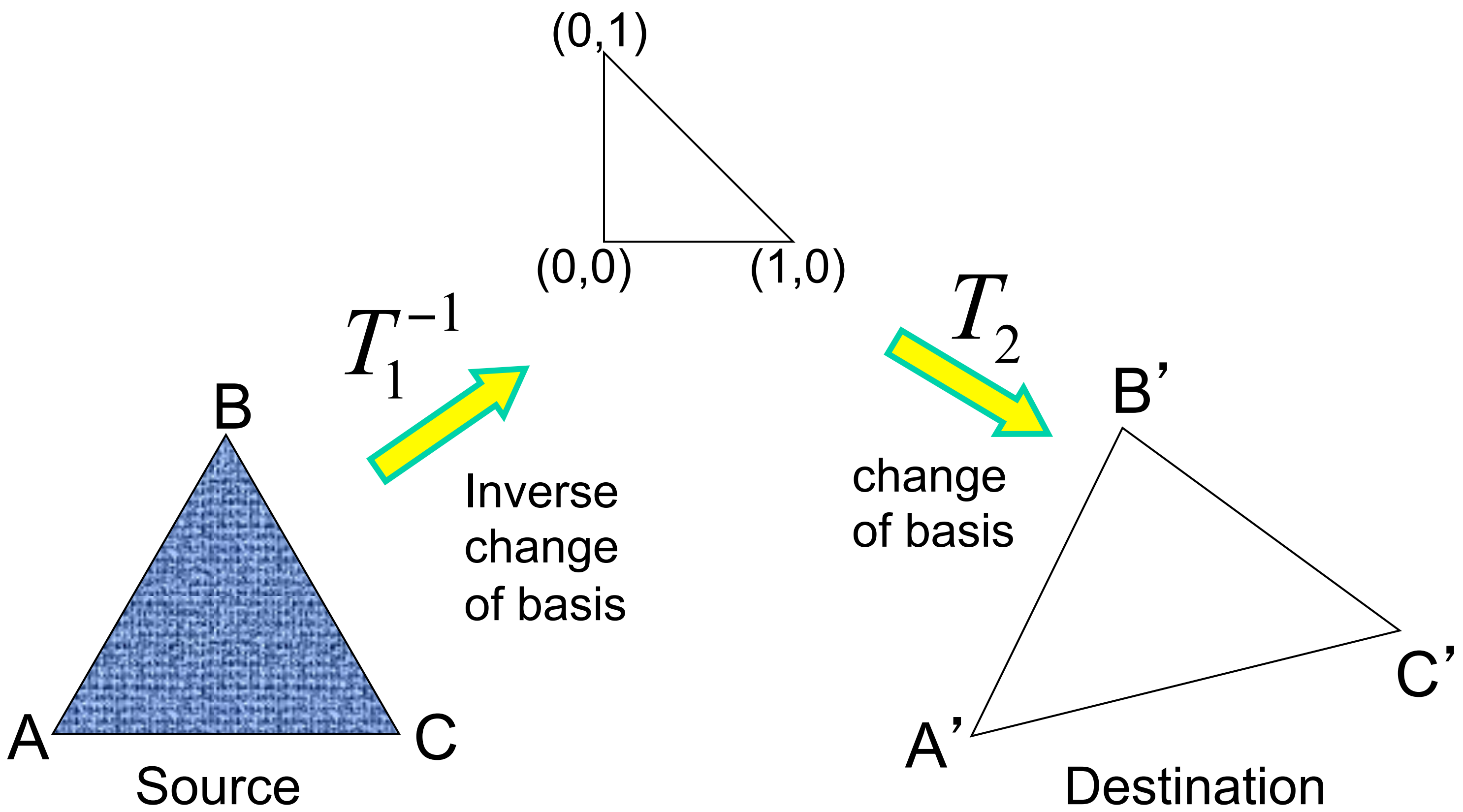
Need to find transform T to transfer all pixels from one to the other.

What kind of transformation is T?

How can we compute the transformation matrix:

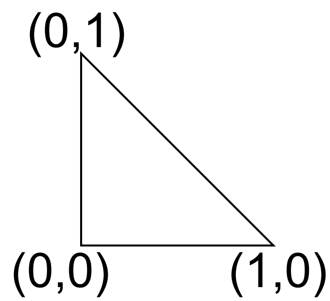
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

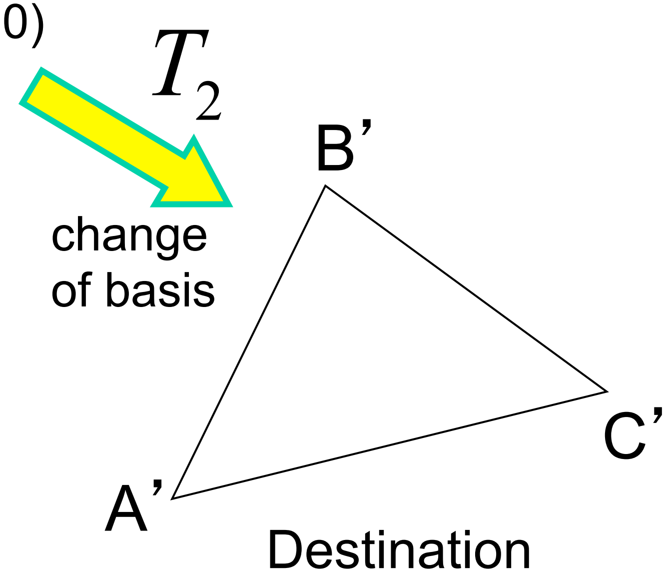
HINT: warping triangles



Don't forget to move the origin too!

Triangle warping…





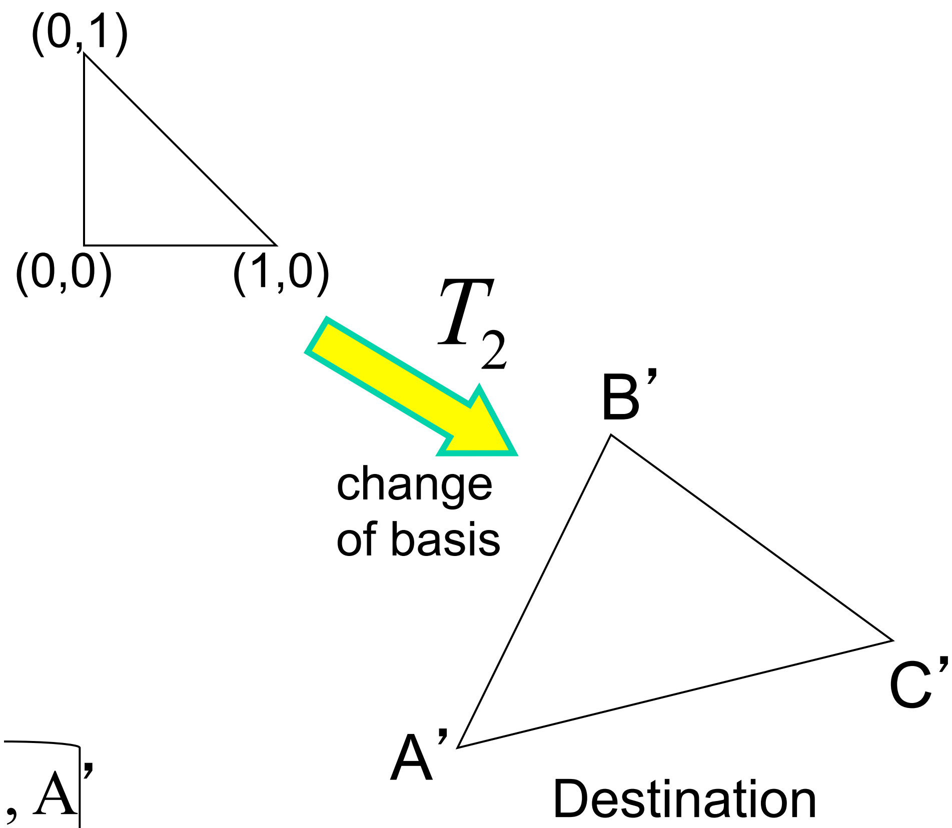
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} a \\ c \end{matrix}} & \boxed{\begin{matrix} b \\ d \end{matrix}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Transformed **x**-axis
↓
(C' -A')

Transformed **y**-axis
↓
(B' -A')

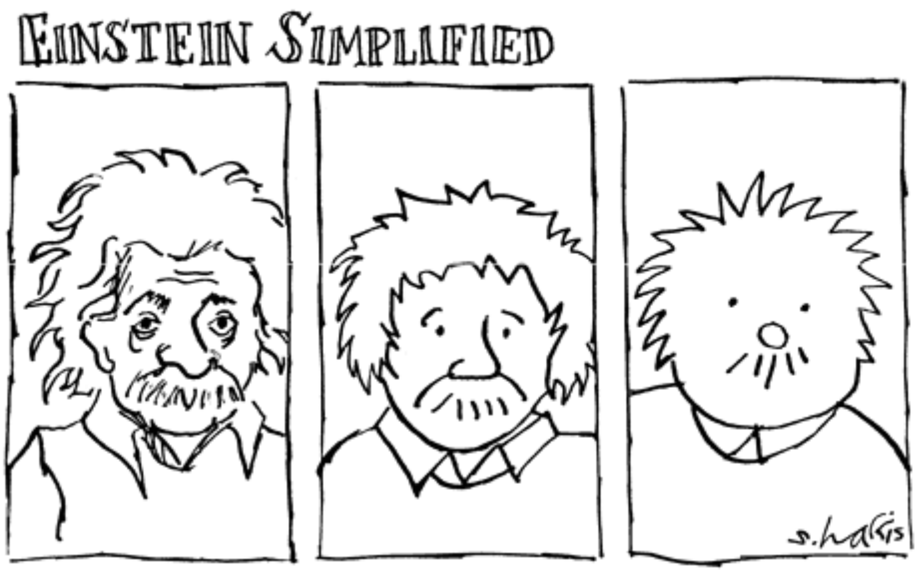
Shift = A'

Triangle warping…

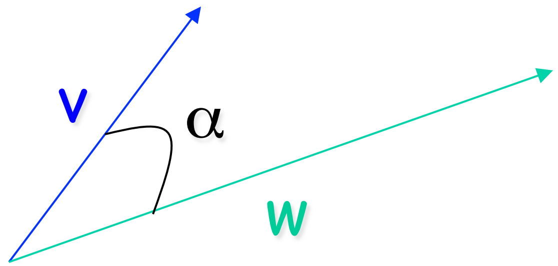


$$T_2 = \begin{bmatrix} C' - A', & B' - A', & A' \\ 0, & 0 & , 1 \end{bmatrix}$$

Linear Algebra Simplified

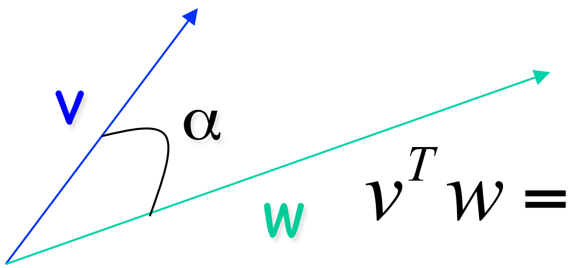


Inner (dot) Product



$$v^T w = (x_1, x_2, x_3)\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = x_1 y_1 + x_2 . y_2 + x_3 . y_3$$

Inner (dot) Product

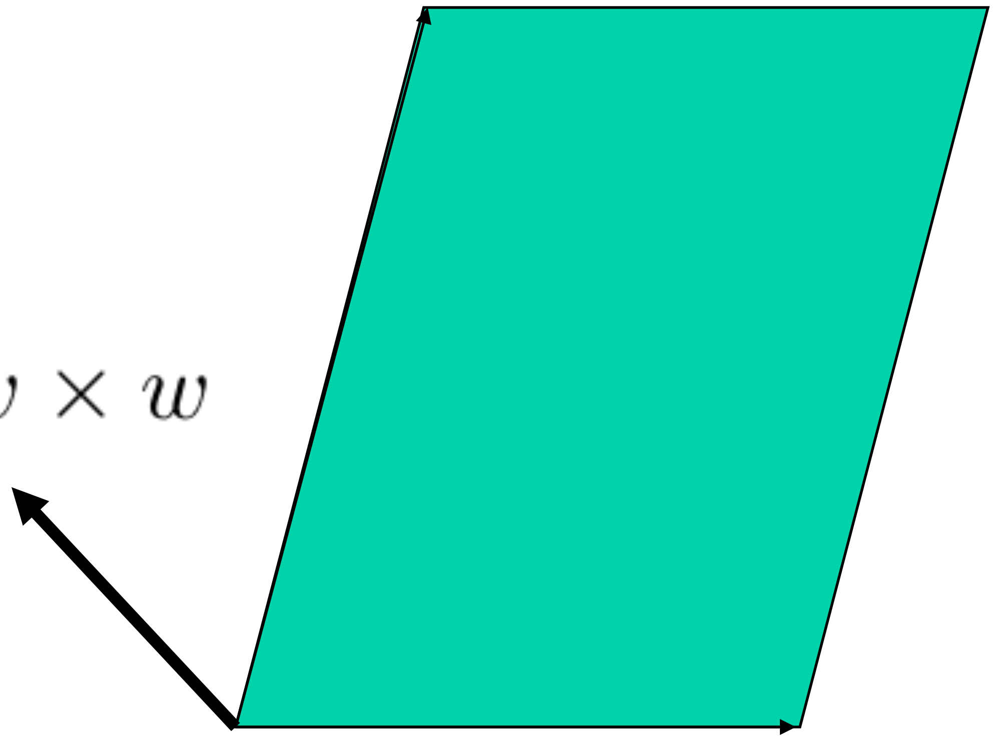


$$v^T w = (x_1, x_2, x_3)\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = x_1 y_1 + x_2 . y_2 + x_3 . y_3$$

$$v^T w = 0 \Leftrightarrow v \perp w$$

Cross product

$$x = v \times w$$



X is normal to
both v and w

$$x = det[(i, j, k); v; w]$$

Cross Product

$$
\begin{aligned}
v \times w &= \boxed{\begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}} \\
&= (x_2 y_3 - x_3 y_2) i \\
&\quad + (x_3 y_1 - x_1 y_3) j \\
&\quad + (x_1 y_2 - x_2 y_1) k \\
&= (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)^T
\end{aligned}
$$

Cross Product

$$
\begin{aligned}
v \times w &= \begin{array}{c} + \\ \begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \end{array} \begin{array}{ccc} - & & \\ i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{array} \\
&= (x_2 y_3 - x_3 y_2) i \\
&\quad + (x_3 y_1 - x_1 y_3) j \\
&\quad + (x_1 y_2 - x_2 y_1) k \\
&= (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)^T
\end{aligned}
$$

Cross Product

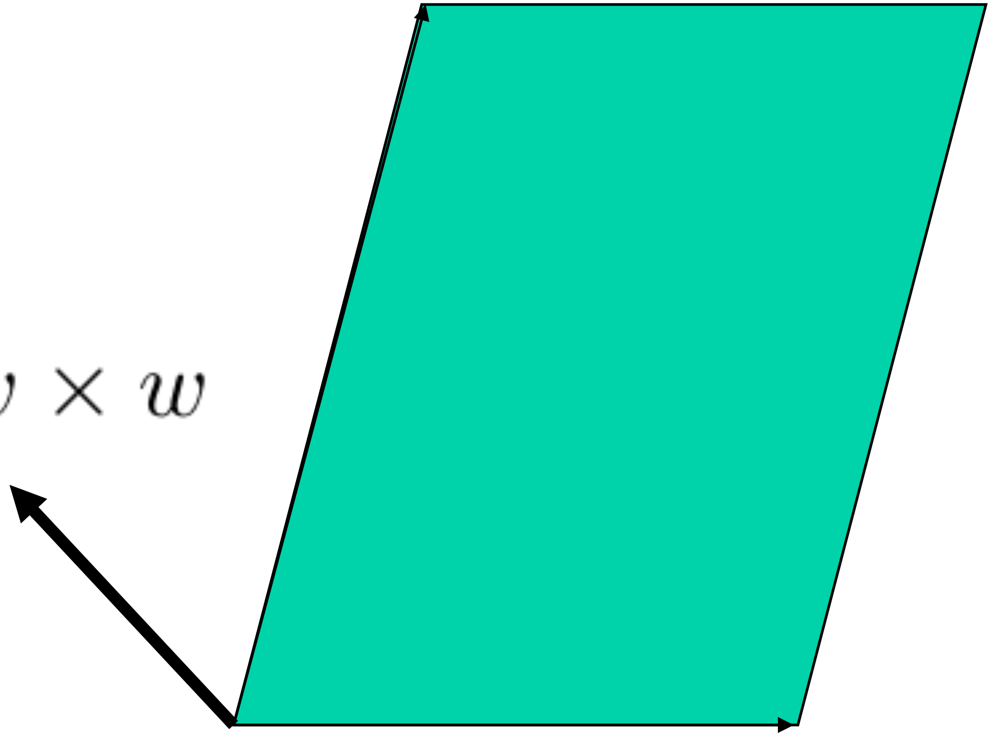
$$
\begin{aligned}
&\qquad\qquad\quad + \qquad\qquad\qquad\qquad\quad - \\
v \times w &= \begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \begin{matrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{matrix} \\
&= (x_2 y_3 - x_3 y_2) i \\
&\quad + (x_3 y_1 - x_1 y_3) j \\
&\quad + (x_1 y_2 - x_2 y_1) k \\
&= (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)^T
\end{aligned}
$$

Cross Product

$$
\begin{aligned}
v \times w &= \left|\begin{array}{ccc} i & j & \overset{+}{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{array}\right| \begin{array}{ccc} i & j & \overset{-}{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{array} \\
&= (x_2 y_3 - x_3 y_2) i \\
&\quad + (x_3 y_1 - x_1 y_3) j \\
&\quad + (x_1 y_2 - x_2 y_1) k \\
&= (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)^T
\end{aligned}
$$

$$v \perp v \times w$$
$$w \perp v \times w$$
$$v^T(v \times w) = 0$$
$$w^T(v \times w) = 0$$

$$x = v \times w$$



Example

$$
\begin{aligned}
v &= (1,2,4)^T \\
w &= (3,1,2)^T \\
v \times w &= (2 \times 2 - 4 \times 1, 4 \times 3 - 1 \times 2, 1 \times 1 - 2 \times 3) \\
&= (0, 10, -5)^T \\
v^T(v \times w) &= 1 \times 0 + 2 \times 10 + 4 \times (-5) \\
&= 0 \\
w^T(v \times w) &= 3 \times 0 + 1 \times 10 + 2 \times (-5) \\
&= 0
\end{aligned}
$$

Lines and Points

Line Representation

- a line is $\rho = x\cos\theta + y\sin\theta$

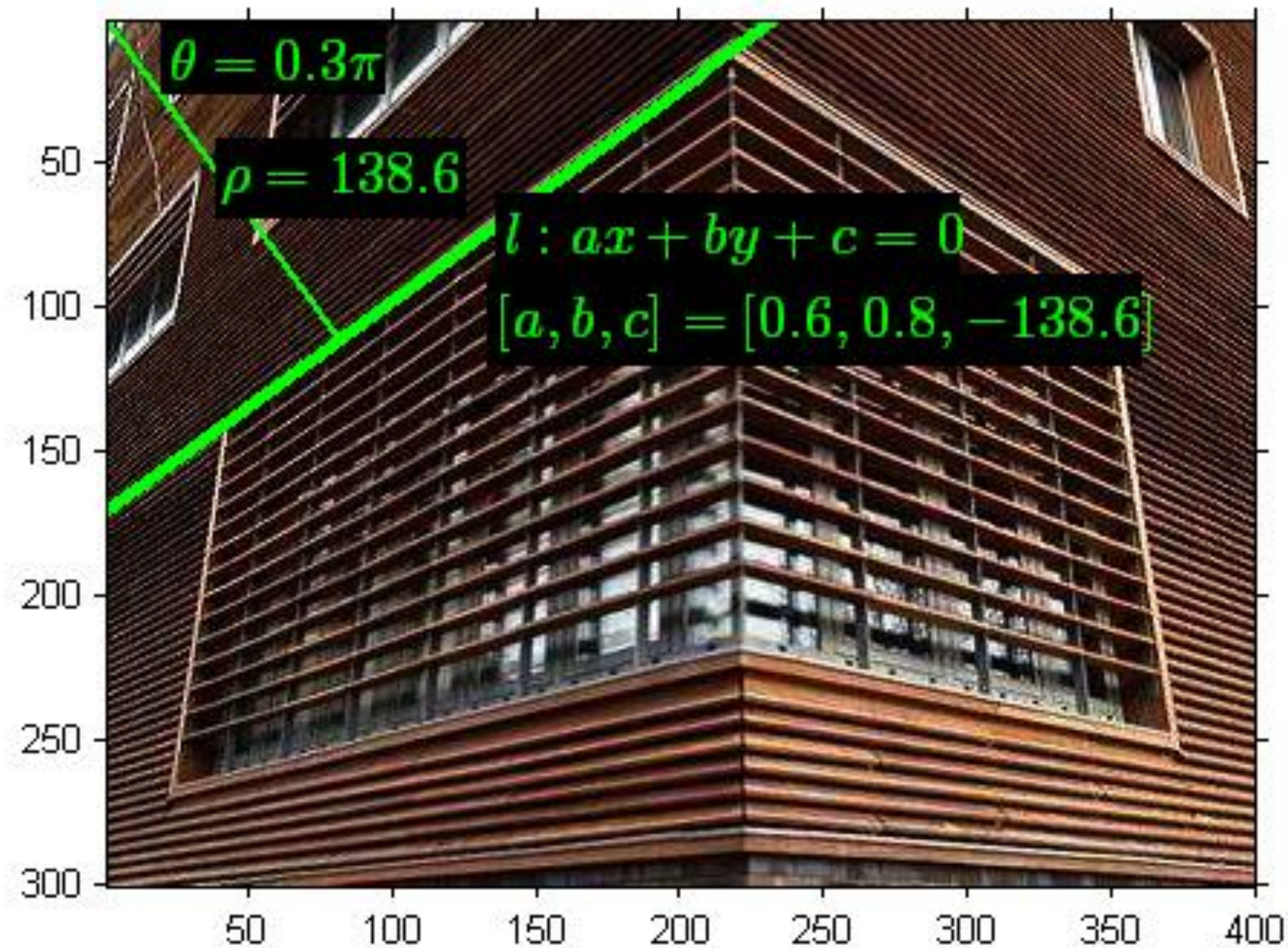
- ρ is the distance from the origin to the line
- θ is the norm direction of the line

- It can also be written as

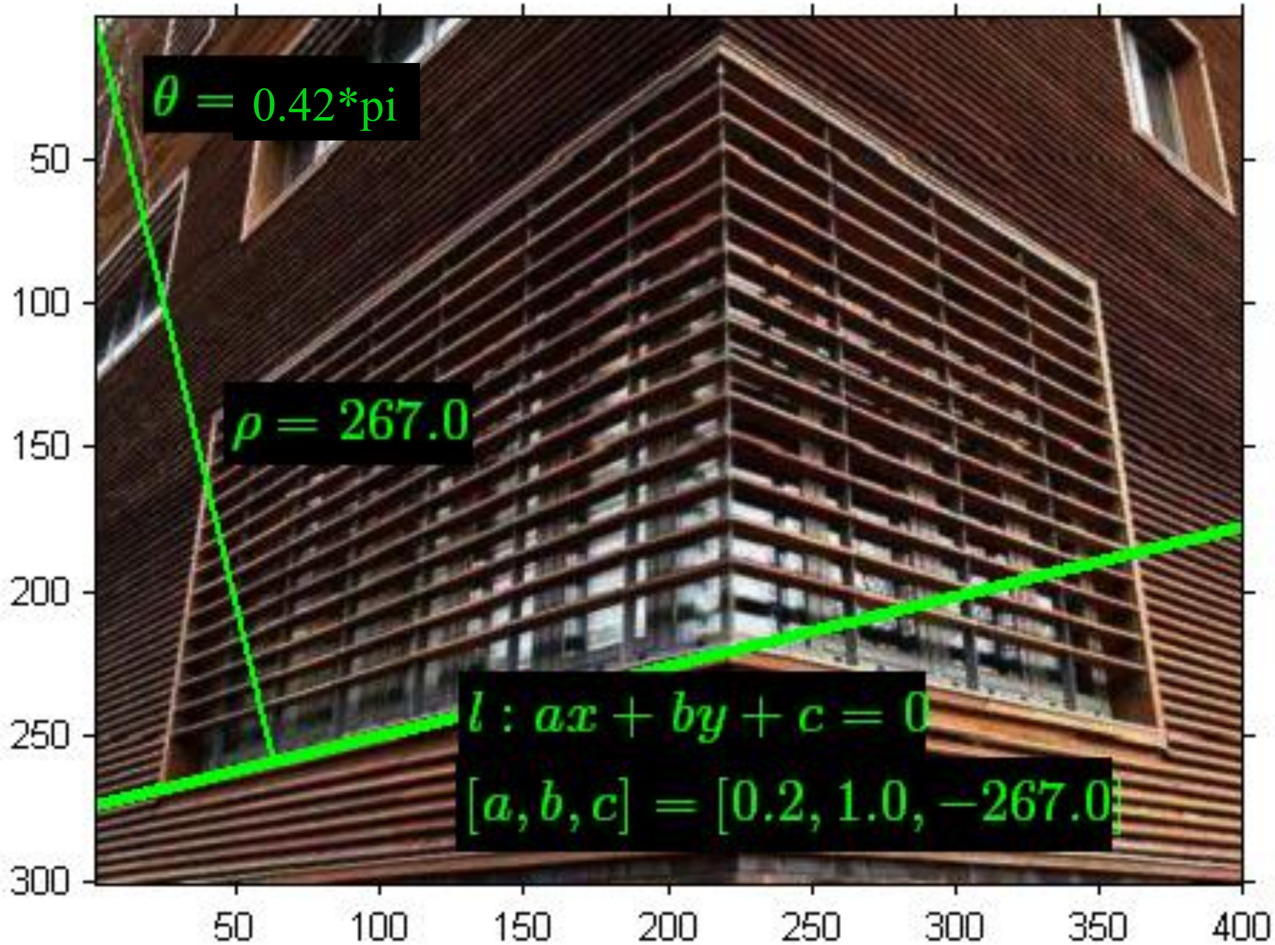
$$ax + by + c = 0;$$

$$\begin{aligned}
\cos\theta &= \frac{a}{\sqrt{a^2+b^2}} \\
\sin\theta &= \frac{b}{\sqrt{a^2+b^2}} \\
\rho &= -\frac{c}{\sqrt{a^2+b^2}}
\end{aligned}$$

Example of Line



Example of Line (2)



Line

Homogeneous Representation

Homogeneous representation

Line in R^2 $\quad ax + by + c = 0;$

Is represented by a point in : $\quad R^3 \quad (a, b, c)$

But correspondence of line to point is not unique $\quad k(a, b, c)$

We define set of equivalence class of vectors in R^3 - (0,0,0)
As projective space

$$P^2$$

Point

Homogenous representation of point

A point lies on a line: $ax + by + c = 0$

$$(x, y, 1)(a, b, c)^T = 0$$

$$(x, y, 1)l = 0$$

A point in P^2 is defined by the equivalence class of $k(x,y,1)$

$$(u, v, w) \qquad x = \frac{u}{w} \quad y = \frac{v}{w}$$

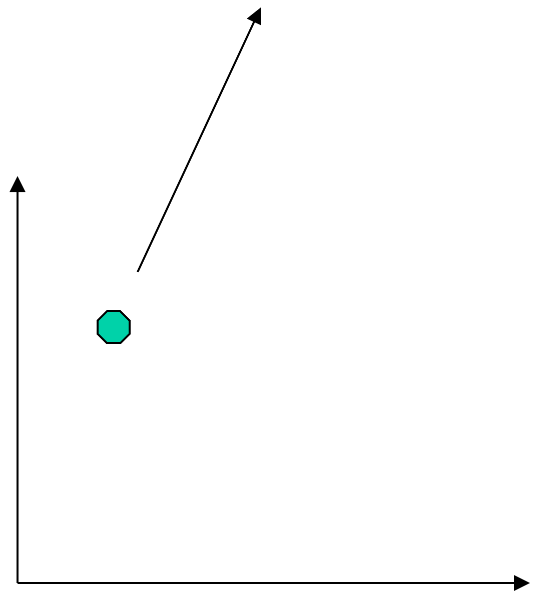
Homogeneous Coordinates

Homogeneous coordinates

- represent coordinates in 2 dimensions with a 3-vector

$$\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{\text{homogeneous coords}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

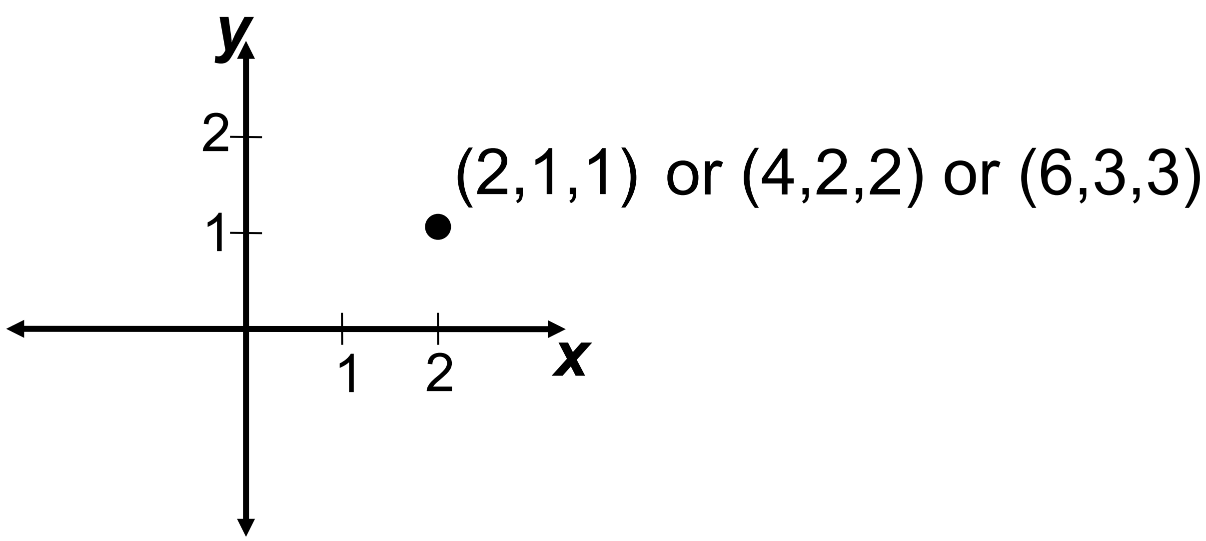
A point:



Homogeneous -> Real Coordinates

divide the third number out:

- (x, y, w) represents a point at location (x/w, y/w)
- (x, y, 0) represents a point at infinity (in direction x,y)
- (0, 0, 0) is not allowed

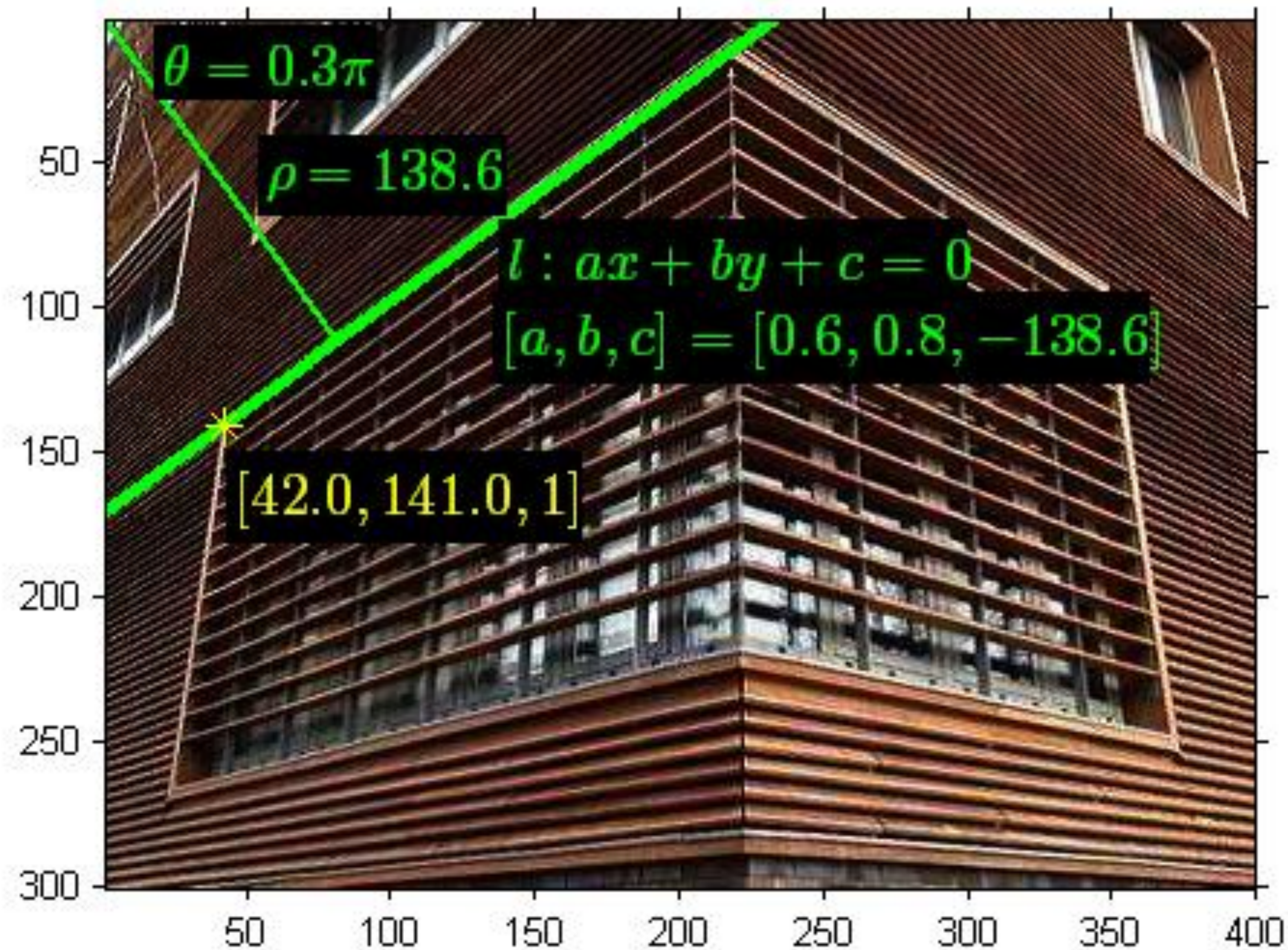


Convenient coordinate system to represent many useful transformations

Example of Point

```
>> 0.6*42+0.8* 141
ans =

   138
```



Line passing two points

Line passing through two points

Two points: $x \quad x'$

Define a line

$$l = x \times x'$$

l is the line passing two points

Proof:

$$x \cdot (x \times x') = 0$$
$$x' \cdot (x \times x') = 0$$

$$x \cdot l = 0$$
$$x' \cdot l = 0$$

Line passing through two points

- More specifically,

$$
\begin{aligned}
l &= x \times x' \\
&= \boxed{\begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ x'_1 & x'_2 & x'_3 \end{vmatrix}} \\
&= (x_2 x'_3 - x_3 x'_2) i \\
&\quad + (x_3 x'_1 - x_1 x'_3) j \\
&\quad + (x_1 x'_2 - x_2 x'_1) k \\
&= (x_2 x'_3 - x_3 x'_2, x_3 x'_1 - x_1 x'_3, x_1 x'_2 - x_2 x'_1)^T
\end{aligned}
$$

Matlab codes

- function l = get_line_by_two_points(x, y)

- x1 = [x(1), y(1), 1]';
- x2 = [x(2), y(2), 1]';
- l = cross(x1, x2);
- l = l / sqrt(l(1)*l(1)+l(2)*l(2));

Example of Line

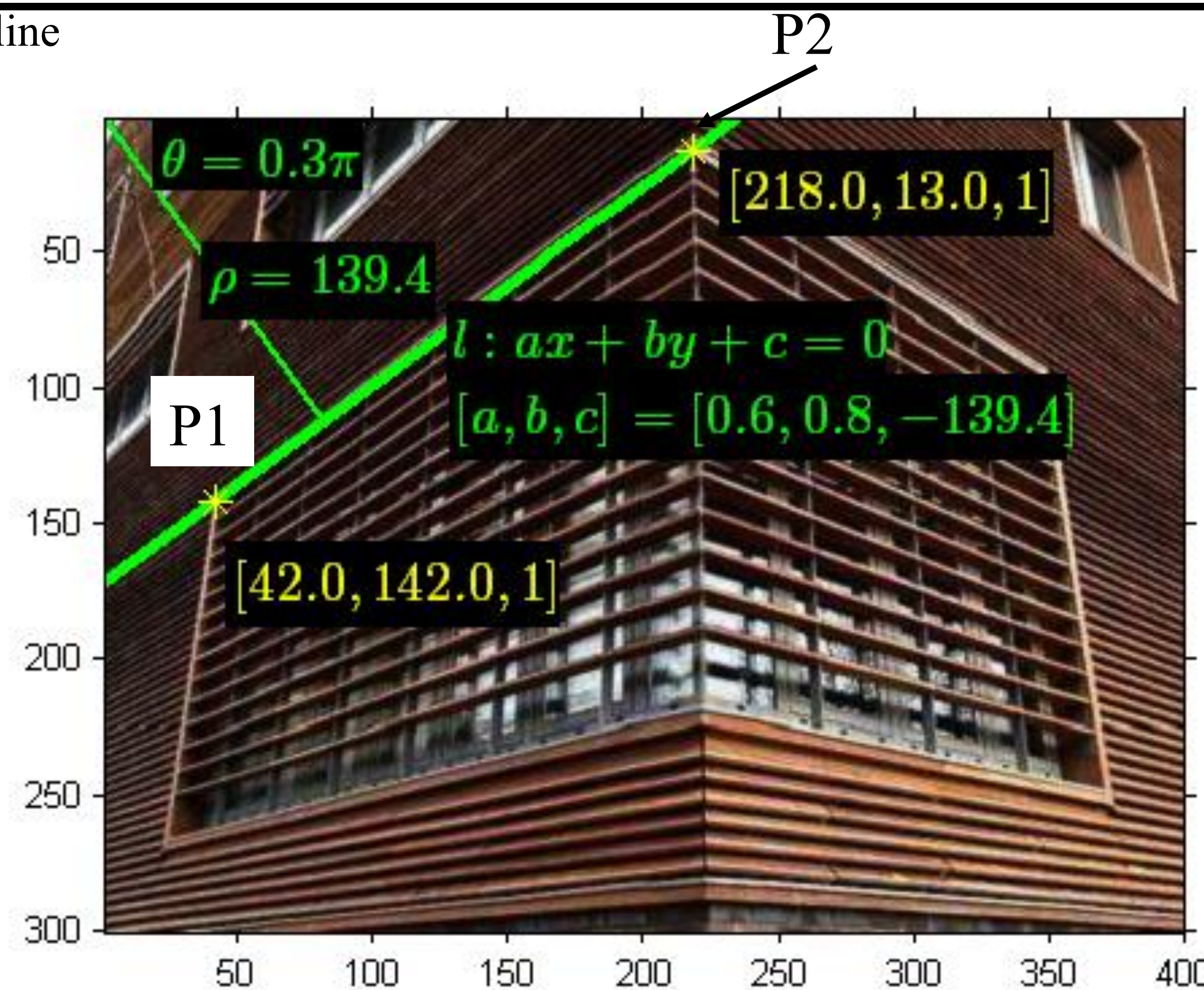
Verify p2 lies on the line

\>> 218*0.6+13*0.8

ans =

141.2000

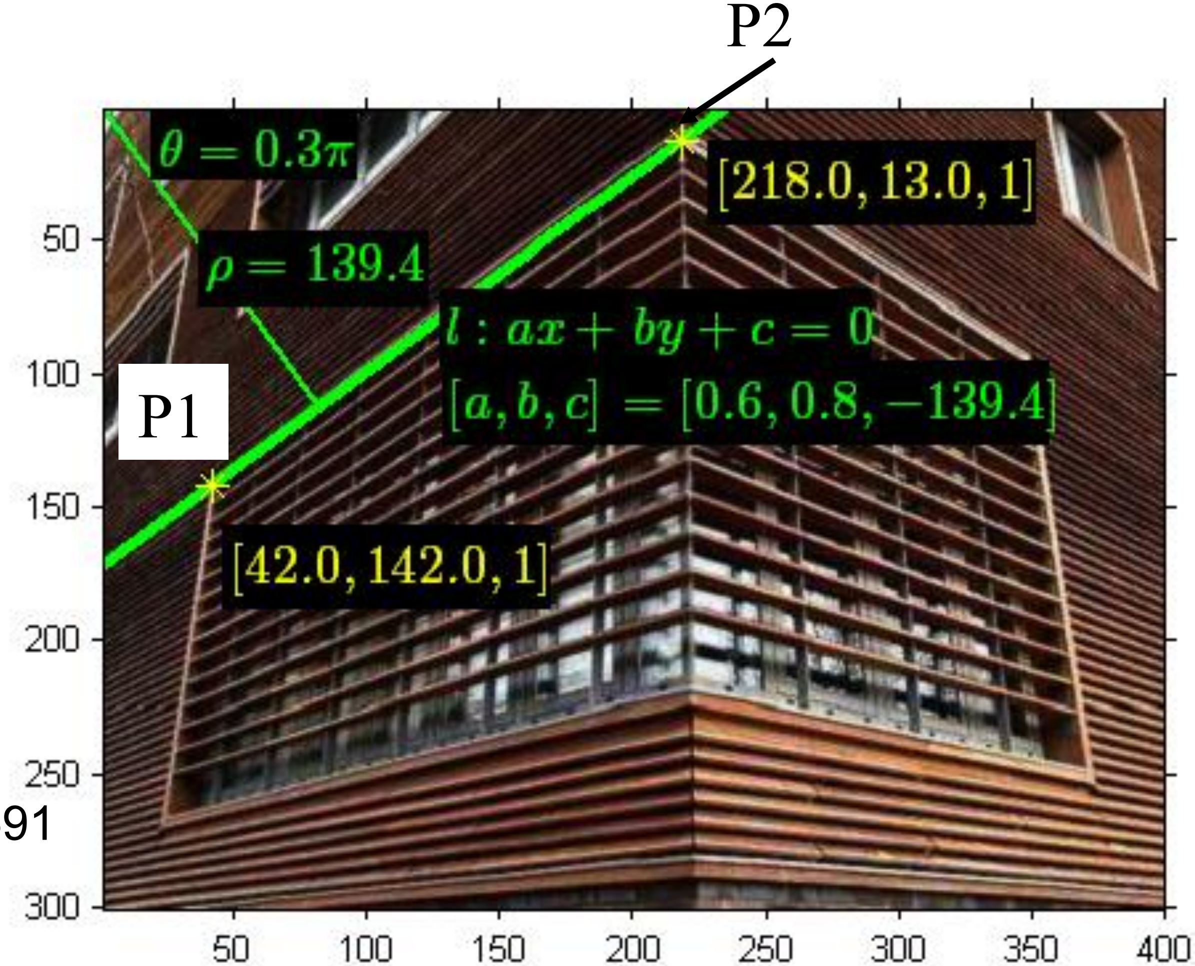
(close)



Example of Line

Test line:

```
>> p1 = [42,142,1];
>> p2=[218,13,1];
>> l = cross(p1,p2)
l =
129   176   -30410
```

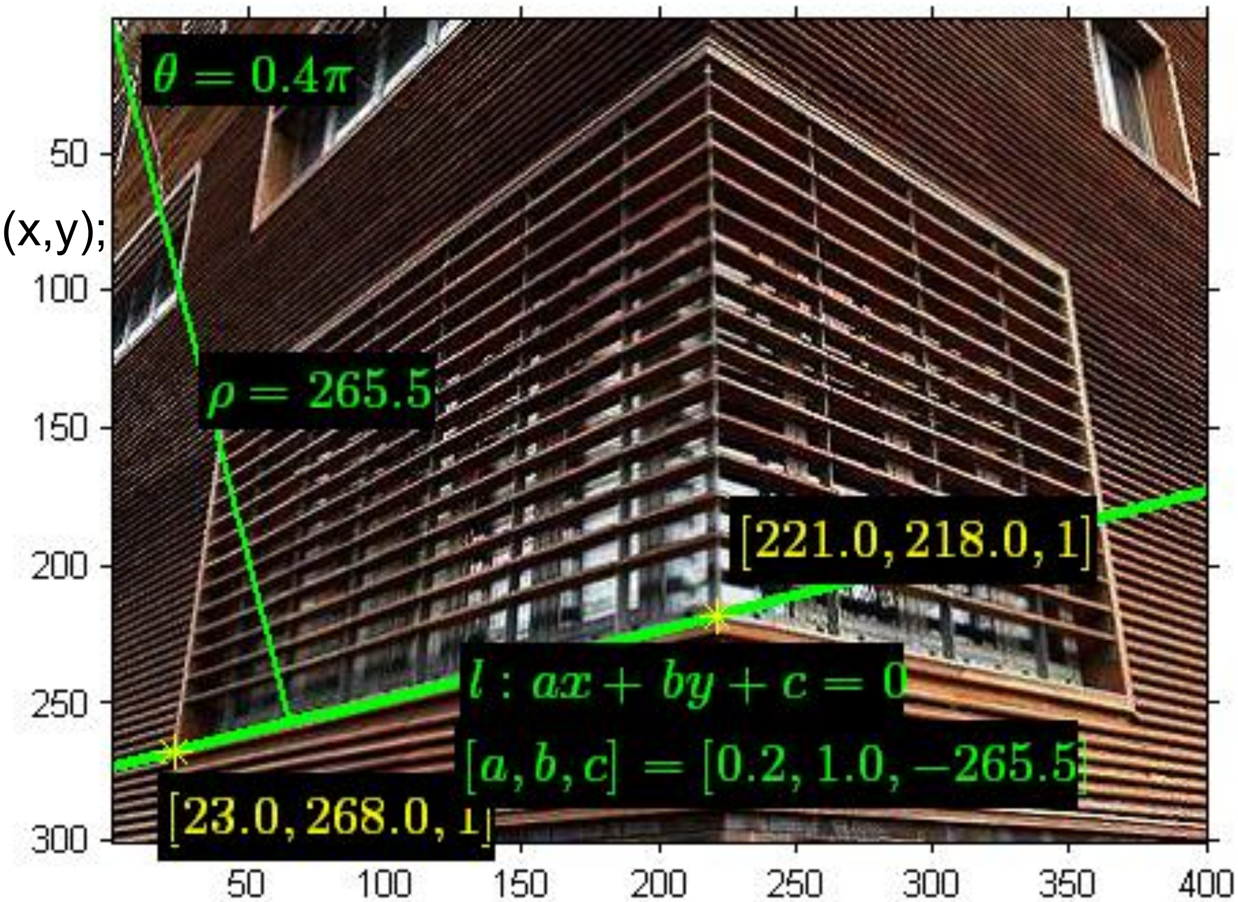



```
>> l = l/sqrt(l(1)^2+l(2)^2)
l =
 0.5912    0.8066 -139.3591
```

```
>> x = [221;23];
>> y = [218;268];
>> l = get_line_by_two_points(x,y);
>> l

l =

   -0.2448
   -0.9696
  265.4744
```



Point passing two lines

Intersection of lines

Given two lines: $l \;\; , \;\; l'$

Define a point

$$x = l \times l'$$

X is the intersection of the two lines

Intersection of lines

- More specifically,

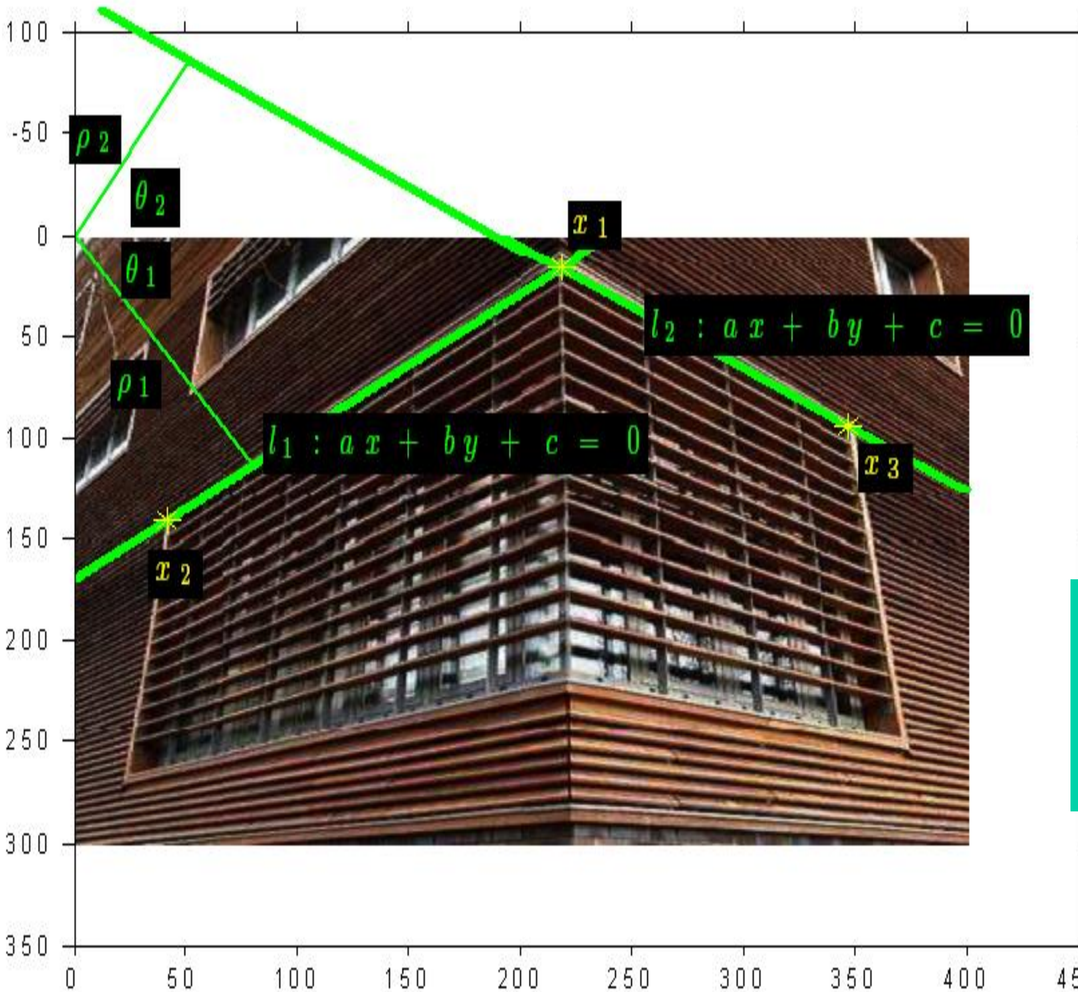
$$
\begin{aligned}
x &= l \times l' \\
& \boxed{= \begin{vmatrix} i & j & k \\ l_1 & l_2 & l_3 \\ l'_1 & l'_2 & l'_3 \end{vmatrix}} \\
&= (l_2 l'_3 - l_3 l'_2) i \\
& \quad + (l_3 l'_1 -3 l_1 l'_3) j \\
& \quad + (l_1 l'_2 - l_2 l'_1) k \\
&= (l_2 l'_3 - l_3 l'_2, l_3 l'_1 - l_1 l'_3, l_1 l'_2 - l_2 l'_1)^T
\end{aligned}
$$

Matlab codes

- function x0 = get_point_by_two_line(l, l1)

- x0 = cross(l, l1);
- x0 = [x0(1)/x0(3); x0(2)/x0(3)];

Example of Lines Intersection



$$\begin{aligned} x_1 &= (218, 16, 1)^T \\ x_2 &= (42, 141, 1)^T \\ x_3 &= (347, 94, 1)^T \end{aligned}$$

$$\begin{aligned} l_1 &= x_1 \times x_2 \\ &= k(-0.58, -0.82, 139.28)^T \\ l_2 &= x_1 \times x_3 \\ &= k(0.52, -0.86, -99.11)^T \end{aligned}$$

$$\begin{aligned} l_1 \times l_2 &= (-199, 98, -14.68, -0.92)^T \\ &= -0.92(218.0, 16.0, 1.0)^T \\ &= kx_1 \end{aligned}$$

Point and Line at infinity

Point at infinity

Example: Consider two *parallel* horizontal lines:

$$x = 1; x = 2;$$

Intersection =

$$det[(i, j, k); (-1, 0, 1); (-1, 0, 2)]$$

$$= (0, 1, 0)$$

Point at infinity in the direction of y

Point at infinity, Ideal points

$$l = (a, b, c) \qquad\qquad l' = (a, b, c')$$

Intersection:

$$\begin{aligned} l \times l' &= l \times l' \\ &= \begin{vmatrix} i & j & k \\ a & b & c \\ a & b & c' \end{vmatrix} \\ &= (bc' - bc, ca - c'a, ab - ab)^T \\ &= (c' - c)(b, -a, 0)^T \end{aligned}$$

Any point $(x_1, x_2, 0)$ is intersection of lines at infinity

Points at infinity

- Under projective transformation,
  - All parallel lines intersects at the point at infinity

$$\text{line } l = (a, b, c)^T \text{ intersects at } (b, -a, 0)^T$$

  - One point at infinity ⇔ one parallel line direction

- Where are the points at infinity in the image plane?
  - To be seen later

Line at infinity

- A line passing all points at infinity:

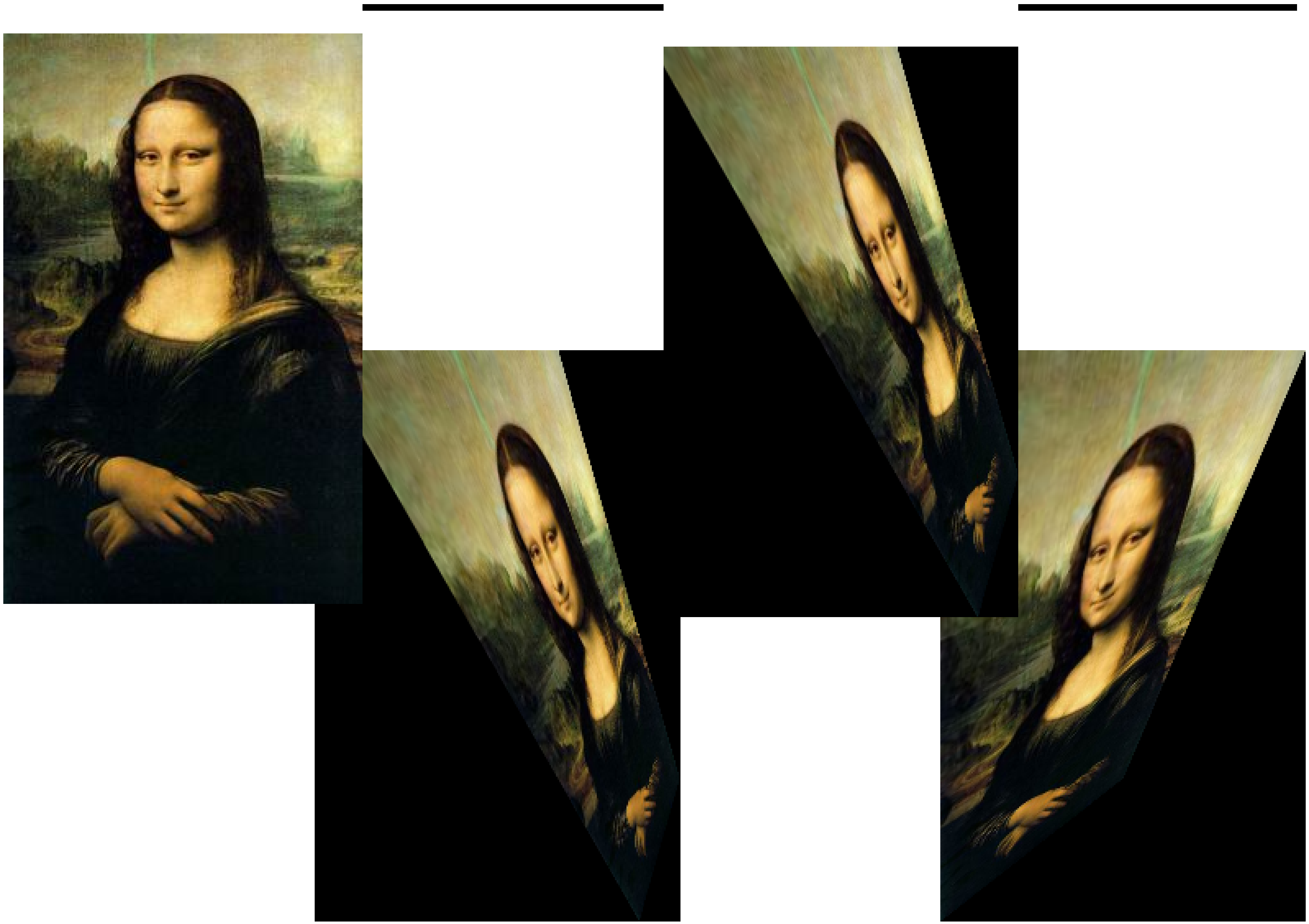
$$l_\infty = (0, 0, 1)^T$$

- Because :

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = 0$$

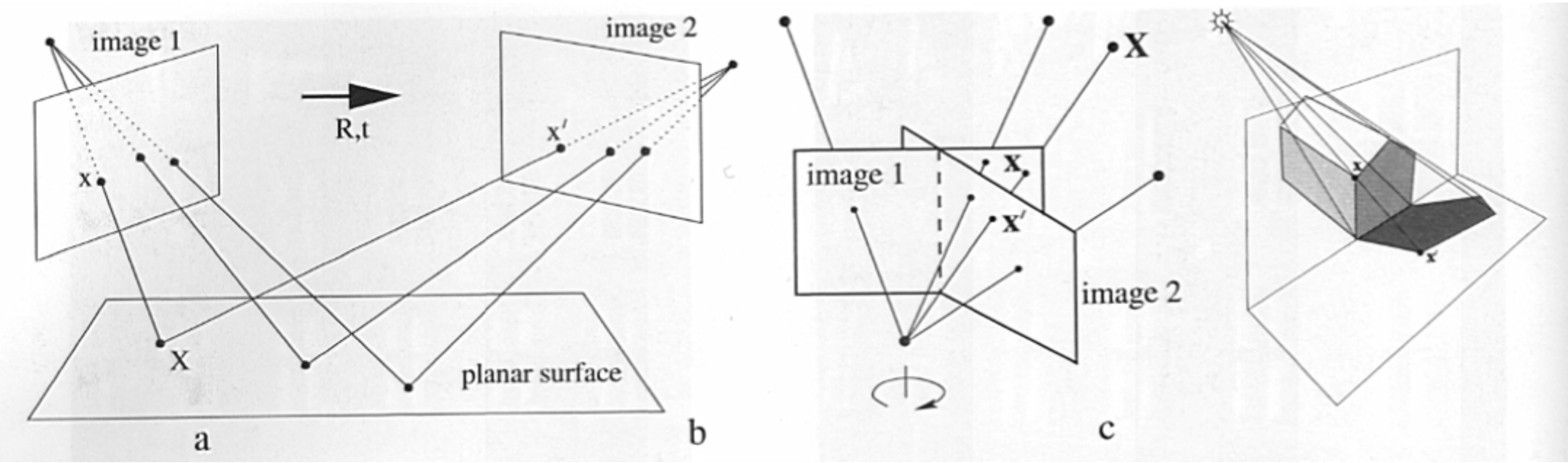
Projective Transformation

From Plane to Plane



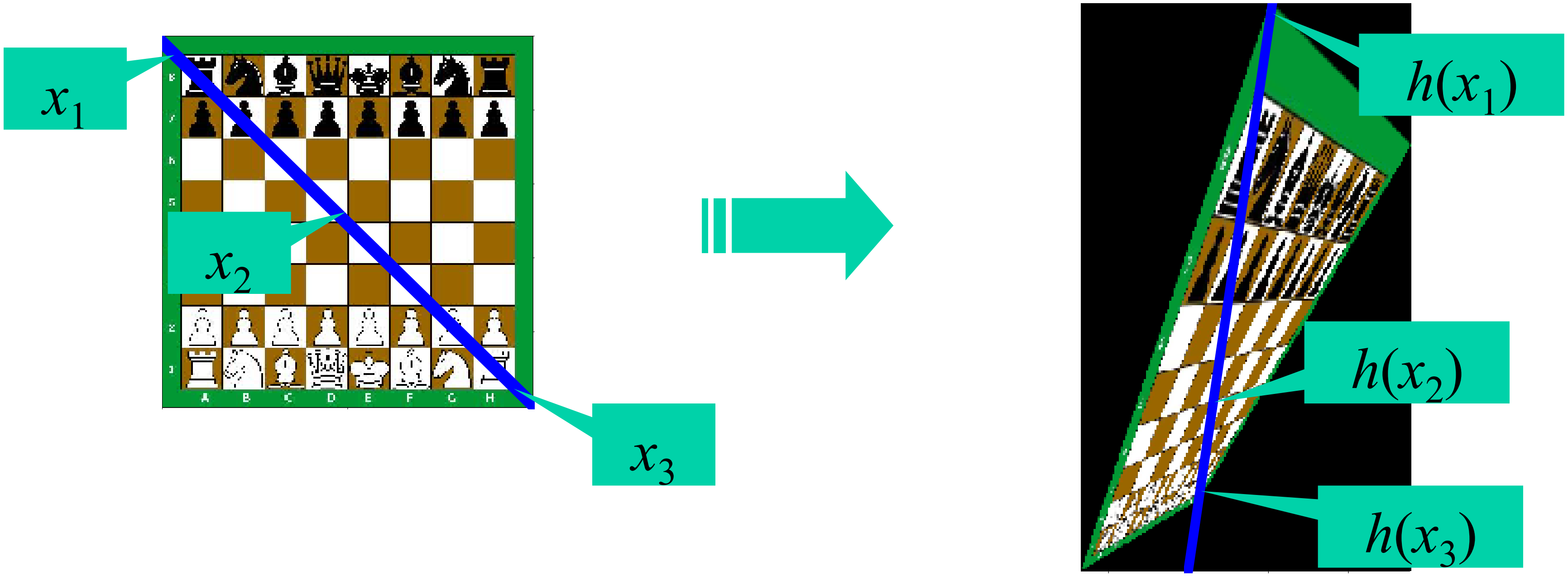
Projective transformation

Goal: study geometry of image projection from one plane to another plane(the image plane).



Facts: 1) parallel lines intersect,
2) circle becomes ellipses,
3) straight line is still straight

Projective transformation



Definition: *Line remains a line!*
Projective transform is an invertible mapping h from P^2 to itself, such that three points x_1, x_2, x_3 lies on a same line iff $h(x_1), h(x_2), h(x_3)$ do.

Projective transformation

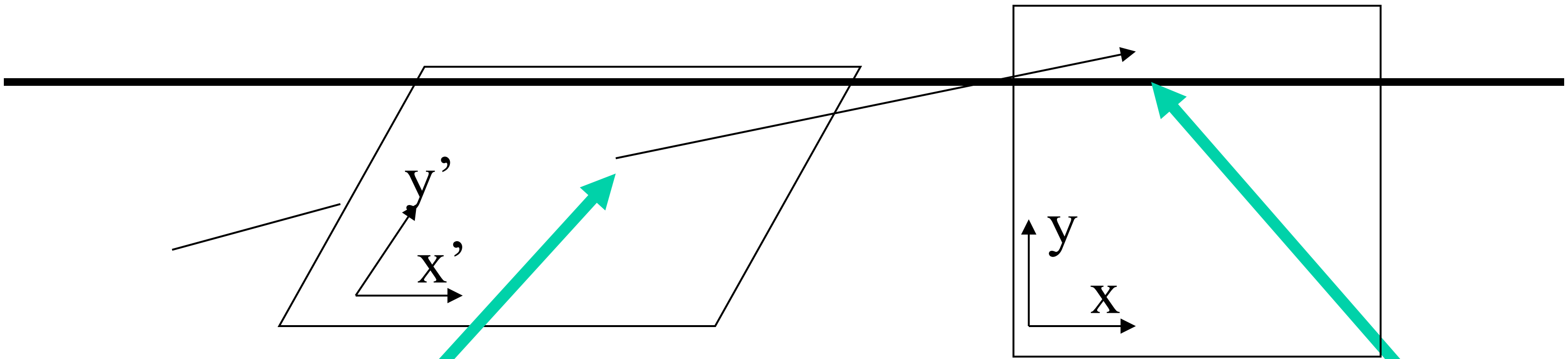
- **Theorem:**

A mapping $h : P^2 \rightarrow P^2$ is a projectivity iff there is a non-singular 3×3 matrix H such that for any point P^2 represented by a vector x it is true that $h(x) = H \cdot x$.

$$H = \begin{bmatrix} \mathrm{h11} & \mathrm{h12} & \mathrm{h13} \\ \mathrm{h21} & \mathrm{h22} & \mathrm{h23} \\ \mathrm{h31} & \mathrm{h32} & \mathrm{h33} \end{bmatrix}$$

Check what happened to x_1, x_2, x_3 lies on line *l*?

Line Mapping: $\boxed{l' = H^{-T} l}$

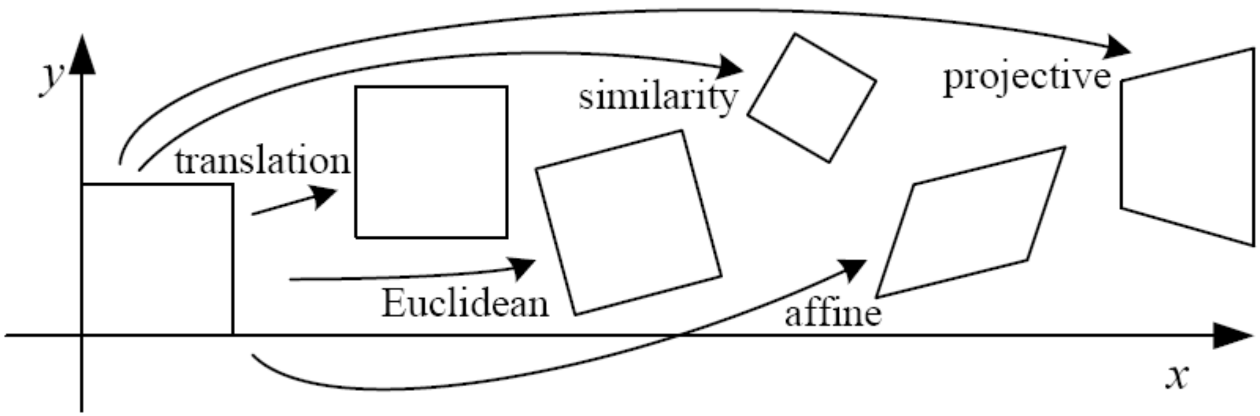


$$\begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \begin{bmatrix} \mathrm{h11} & \mathrm{h12} & \mathrm{h13} \\ \mathrm{h21} & \mathrm{h22} & \mathrm{h23} \\ \mathrm{h31} & \mathrm{h32} & \mathrm{h33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\boxed{x' = \frac{h_{11}x_1 + h_{12}x_2 + h_{13}x_3}{h_{31}x_1 + h_{32}x_2 + h_{33}x_3}}$$

How many independent para? Can we always set h33 = 1?

Classes of 2D projective transformations

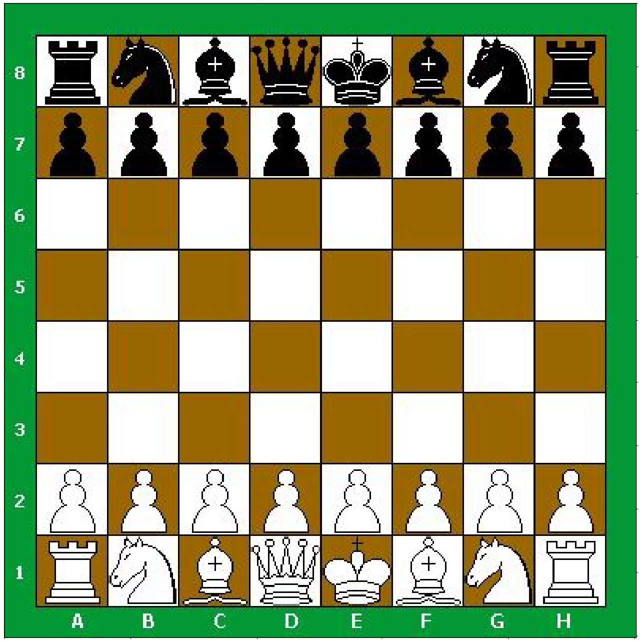


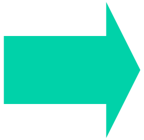
$$\mathrm{H} = \begin{bmatrix} \mathrm{A} & \mathrm{t} \\ \mathbf{v} & v \end{bmatrix}$$

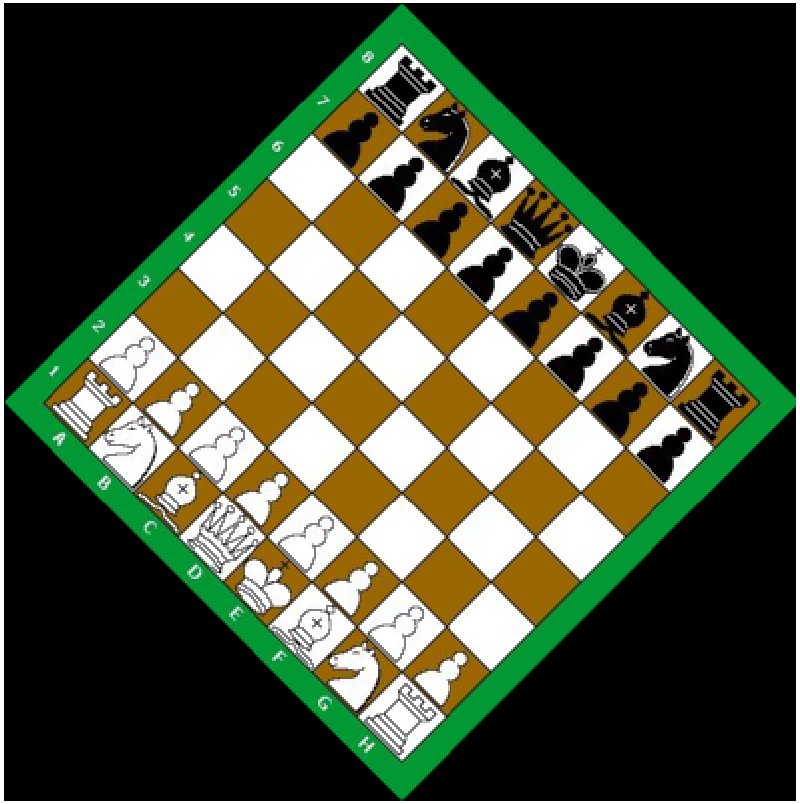
Special case: Similarity Transformation

Similarity

$$H_S = \begin{bmatrix} 2\cos\pi/4 & -2\sin\pi/4 & 1 \\ 2\sin\pi/4 & 2\cos\pi/4 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

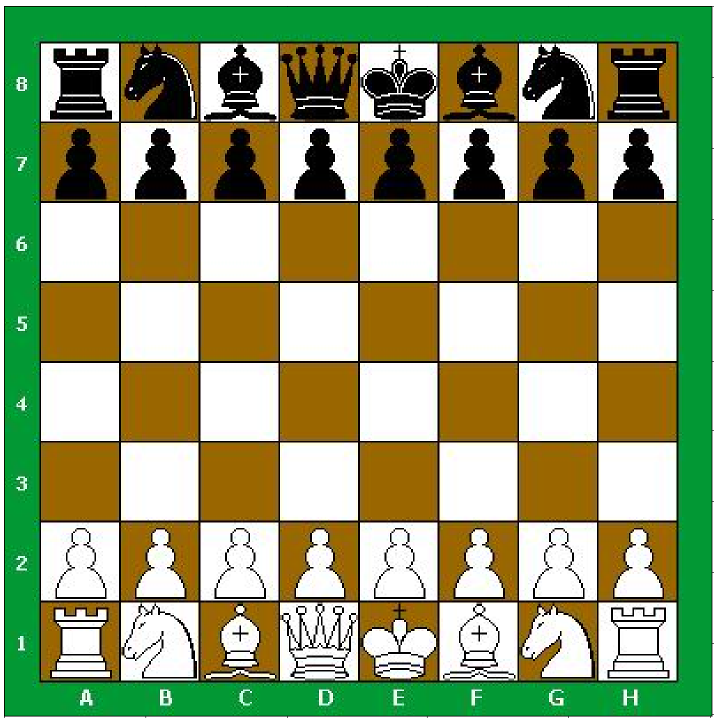


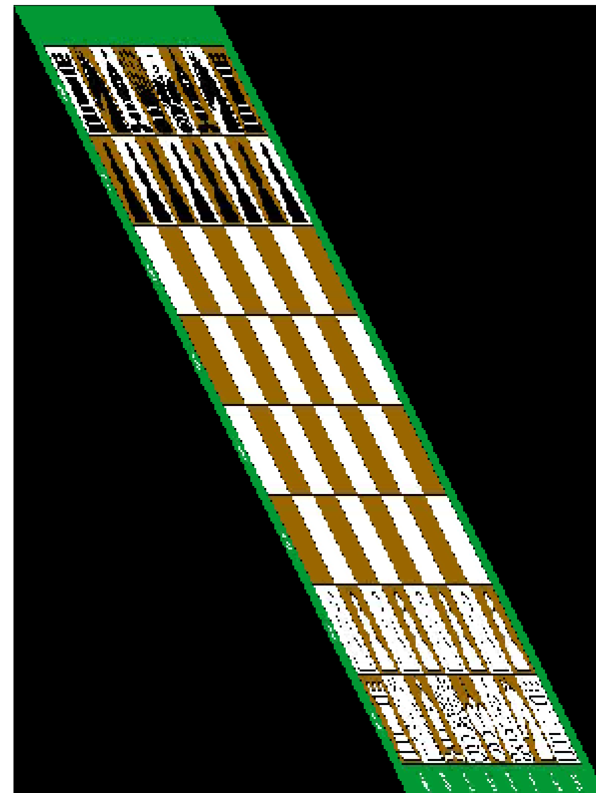




Special Case: Affine Transformation

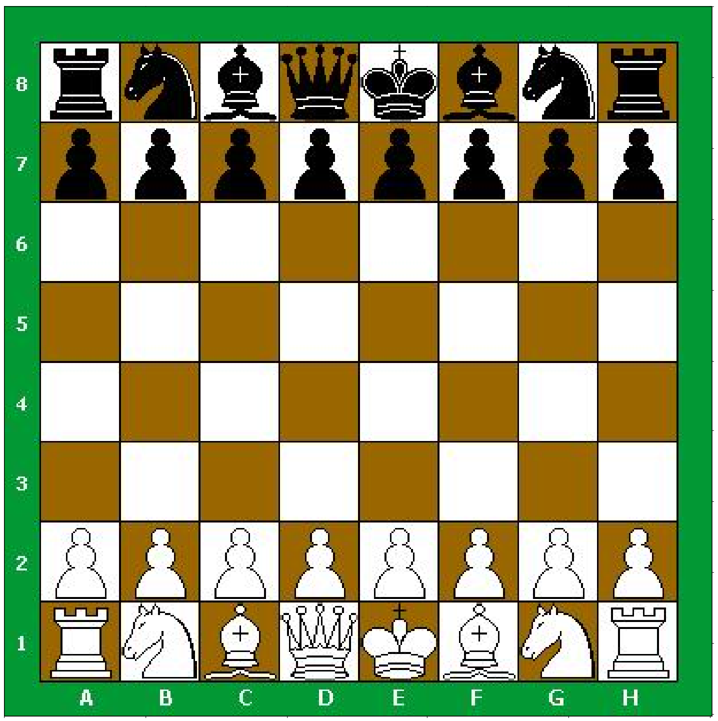
$$H_A = \begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

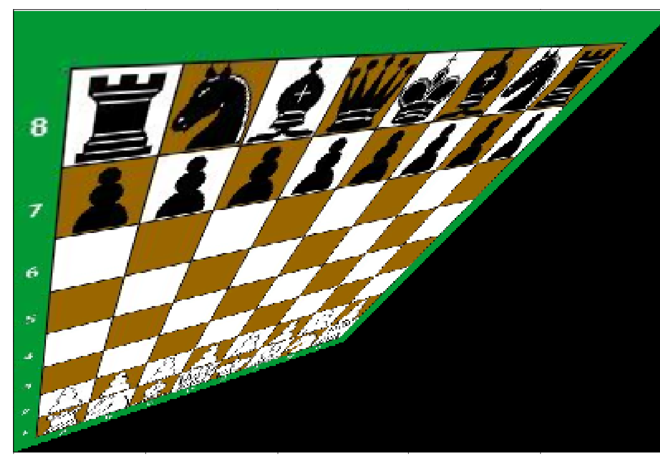


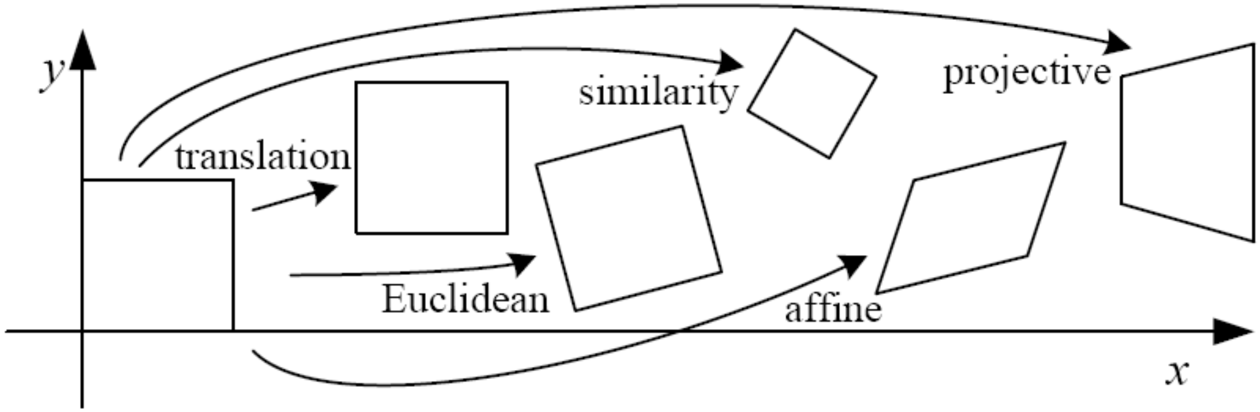


Special case: Projective transformation

$$H_P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$



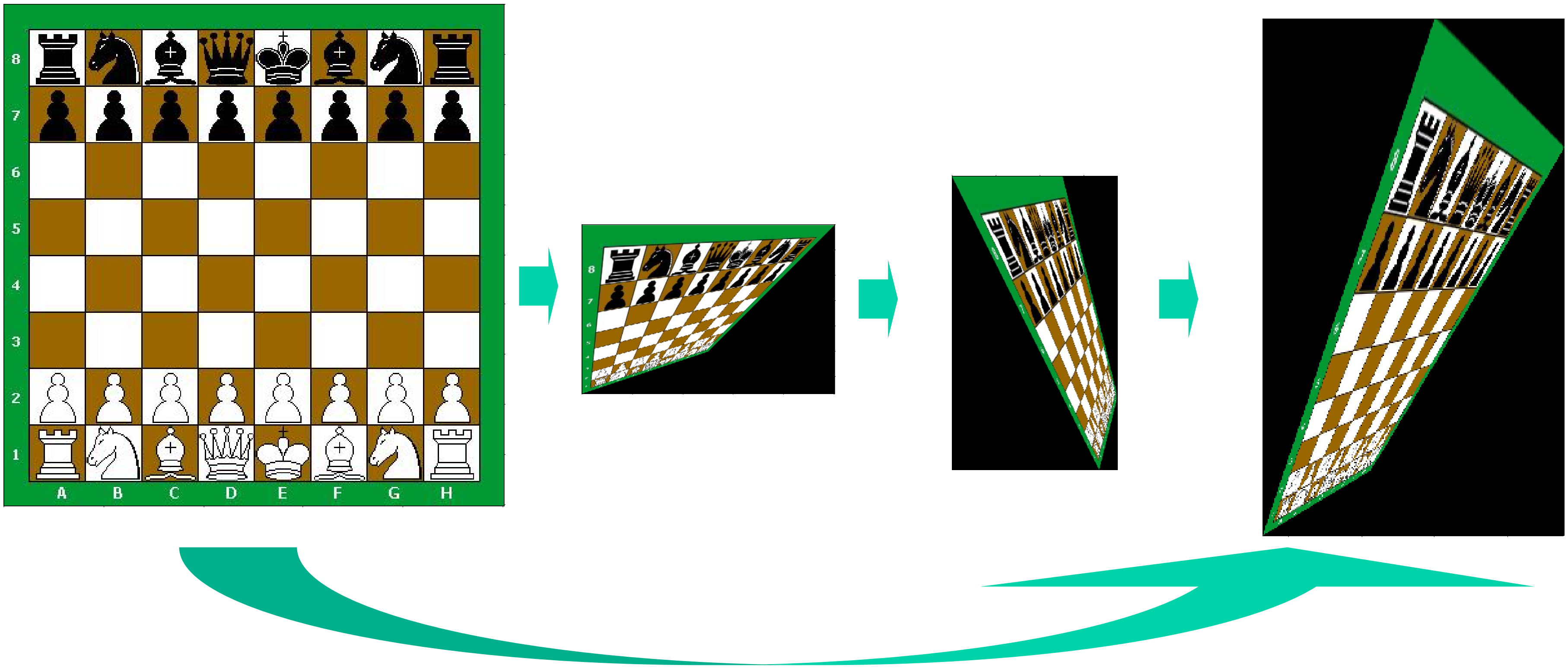




$$H = \begin{bmatrix} A & t \\ \mathbf{v} & v \end{bmatrix} = H_S H_A H_P$$

$$= \begin{bmatrix} R & t \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} K & 0 \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} I & 0 \\ \mathbf{v} & v \end{bmatrix}$$

Example

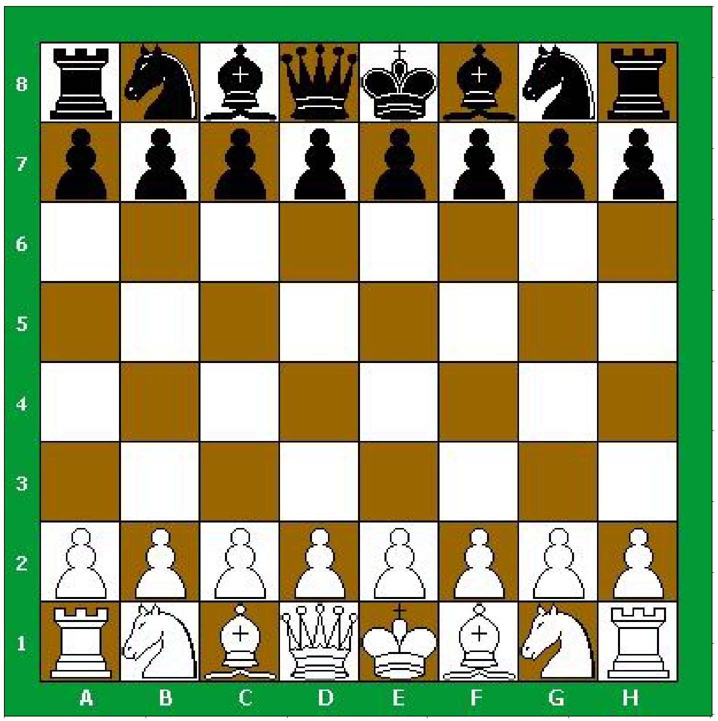


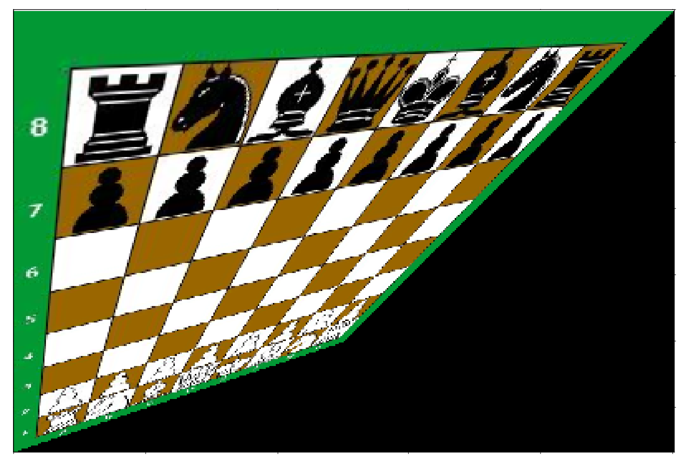
$$H = H_S \times H_A \times H_P = \begin{bmatrix} 1.71 & 0.586 & 1 \\ 2.71 & 8.24 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Example

- Projective

$$H_P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

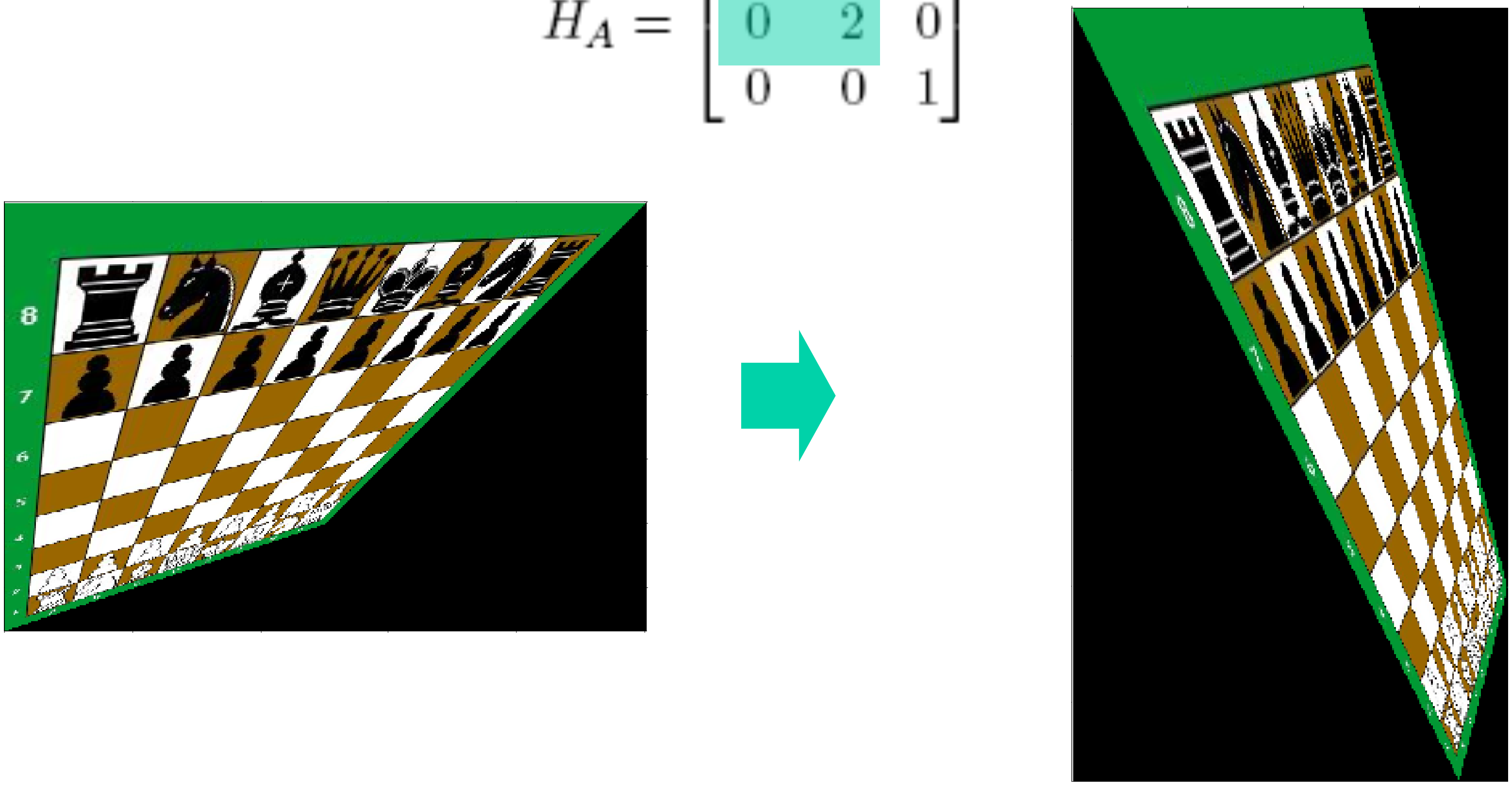




Example

Affine

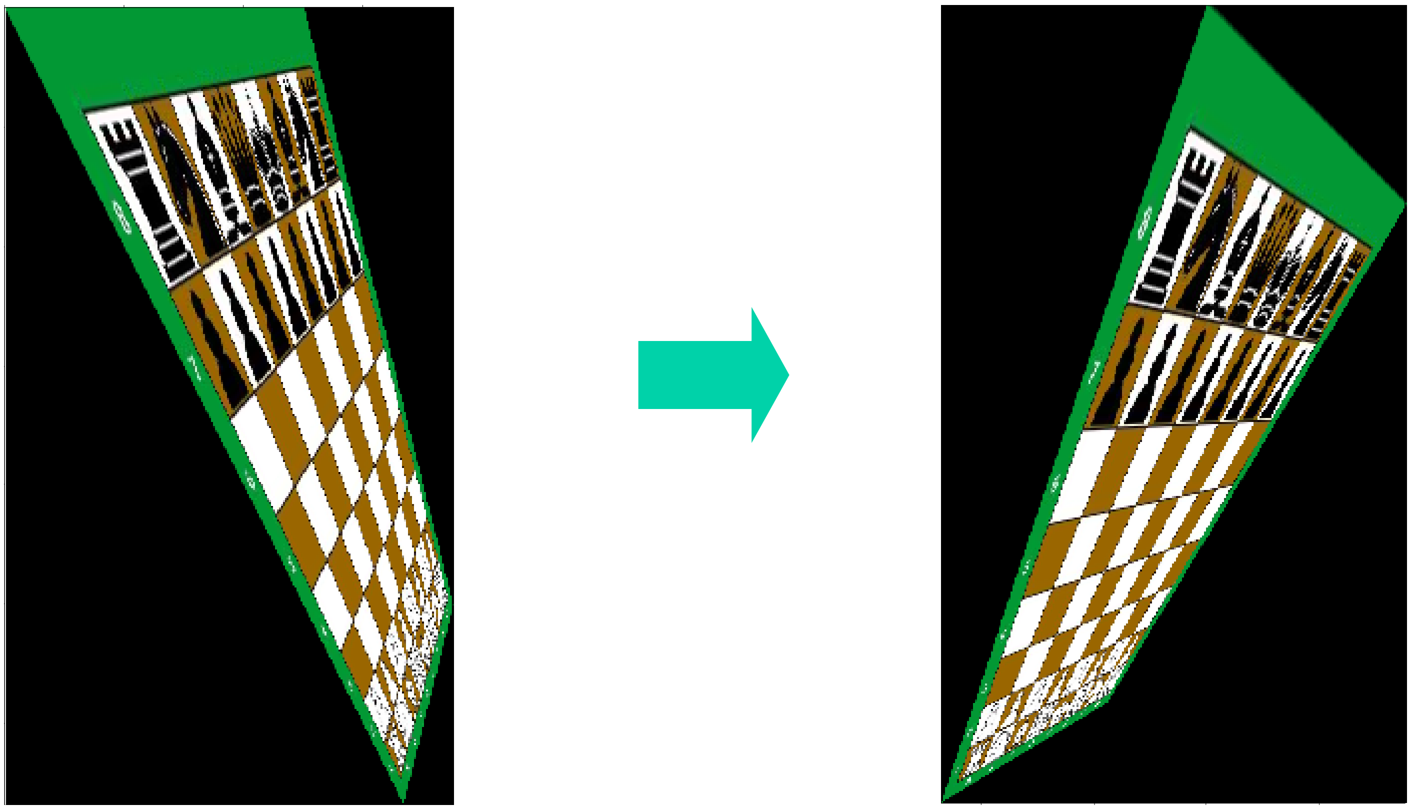
$$H_A = \begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Example

Similarity

$$H_S = \begin{bmatrix} 2\cos\pi/4 & -2\sin\pi/4 & 1 \\ 2\sin\pi/4 & 2\cos\pi/4 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$



Matlab codes

- function img_now = projective_transform(img, H);

- px = [0 1];
- py = [0 1];
- tform = maketform('projective', H');
- [img_now, xdata, ydata]= imtransform(img, tform, 'udata', ...
- px, 'vdata', py, 'size', size(img));

- Extra slides

Points, Lines & Projective Transformation

Lines under Projective Transformation

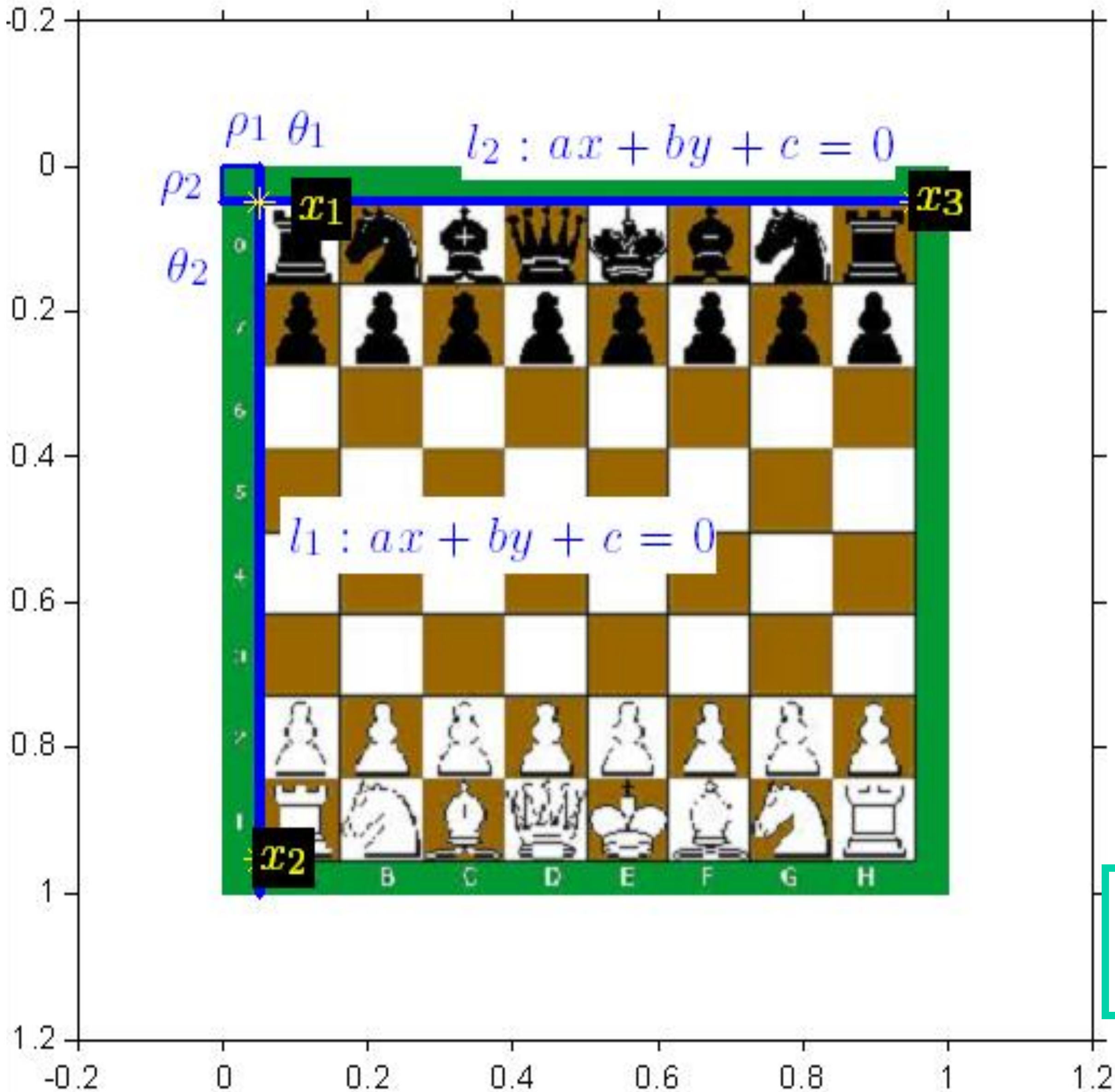
- With homogenous coordinates:
- Point:

$$x' = Hx$$

- Line:

$$l' = H^{-1}l$$

Before transformation



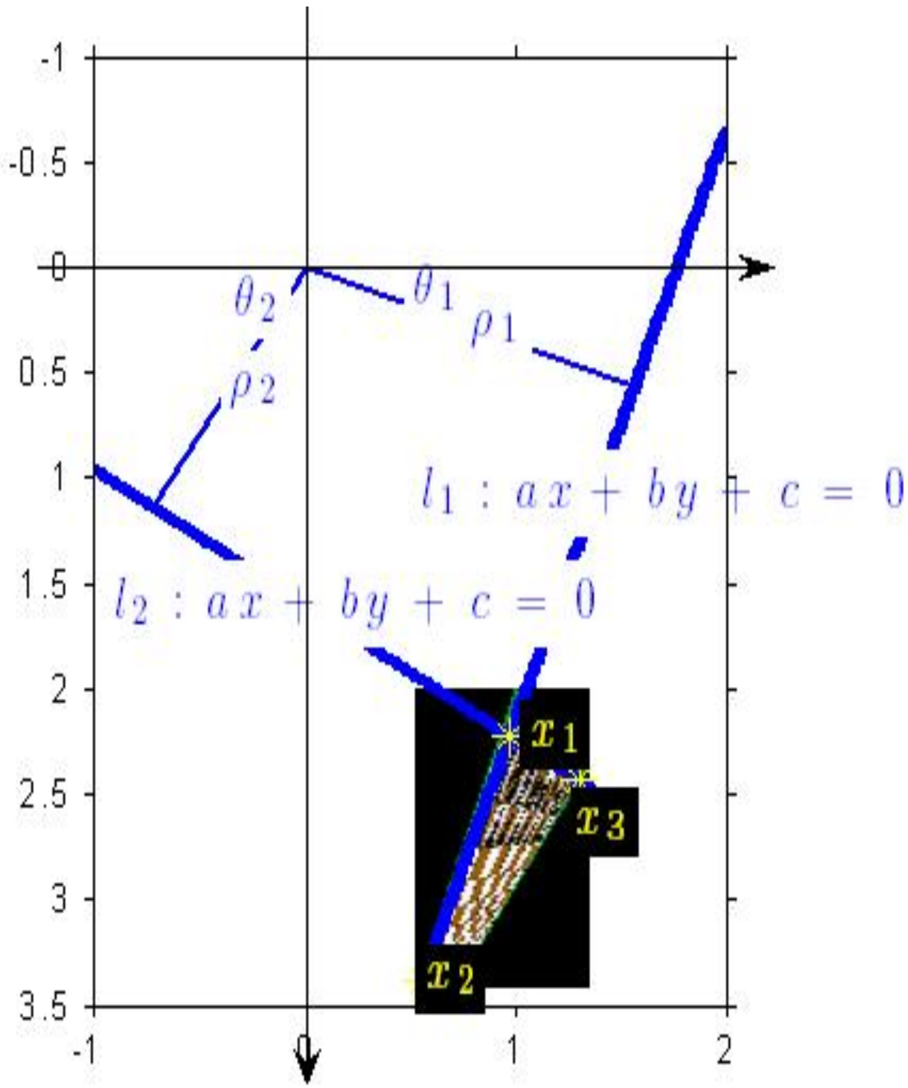
$$
\begin{aligned}
x_1 &= (0.05, 0.05, 1)^T \\
x_2 &= (0.05, 0.95, 1)^T \\
x_3 &= (0.95, 0.05, 1)^T
\end{aligned}
$$

$$
\begin{aligned}
l_1 &= x_1 \times x_2 \\
&= k(-1, 0, 0.05)^T \\
\rho_1 &= 0.05 \\
\theta_1 &= 0
\end{aligned}
$$

$$
\begin{aligned}
l_2 &= x_1 \times x_3 \\
&= k(0, -1, 0.05)^T \\
\rho_2 &= 0.05 \\
\theta_2 &= 0.5\pi
\end{aligned}
$$

$$
\begin{aligned}
l_1 \times l_2 &= (0.05, 0.05, 1.00)^T \\
&= k x_1
\end{aligned}
$$

After transformation



$$
\begin{aligned}
x_1 &= (0.97, 2.22, 1)^T \\
x_2 &= (0.56, 3.38, 1)^T \\
x_3 &= (1.30, 2.43, 1)^T
\end{aligned}
$$

$$
\begin{aligned}
l_1 &= x_1 \times x_2 \\
&= k(-0.94, -0.34, 1.66)^T \\
\rho_1 &= 1.66 \\
\theta_1 &= 0.11\pi
\end{aligned}
$$

$$
\begin{aligned}
l_2 &= x_1 \times x_3 \\
&= k(0.54, -0.84, 1.35)^T \\
\rho_2 &= 1.35 \\
\theta_2 &= 0.68\pi
\end{aligned}
$$

$$
\begin{aligned}
l_1 \times l_2 &= (0.95, 2.17, 0.97)^T \\
&= 0.97(0.97, 2.22, 1.000)^T \\
&= kx_1
\end{aligned}
$$

Just checking…

- Verify:

$$H = \begin{bmatrix} 1.71 & 0.586 & 1 \\ 2.71 & 8.24 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

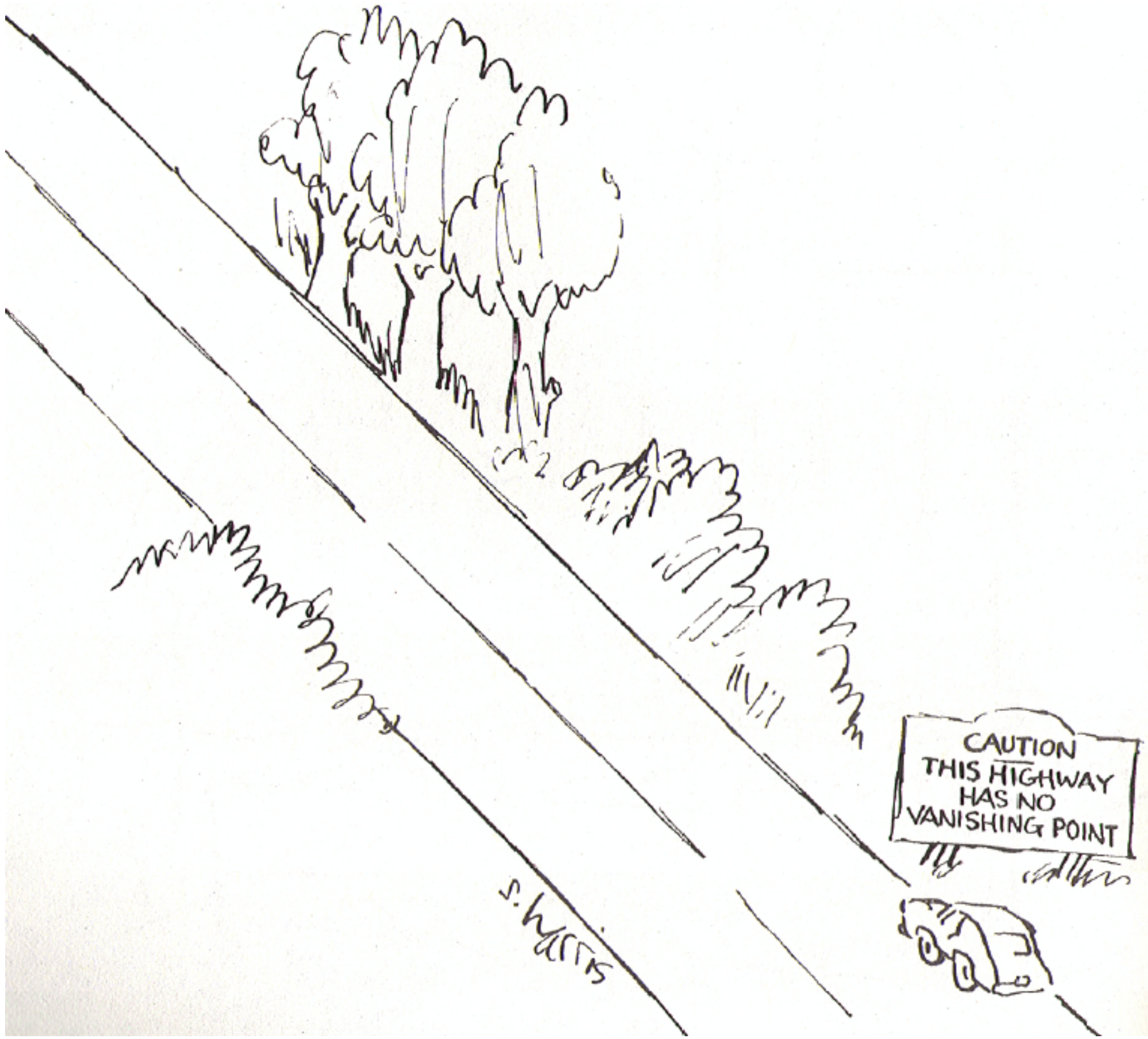
$$H^{-T} = \begin{bmatrix} 1.06 & -0.18 & -0.71 \\ 0.35 & 0.18 & -0.71 \\ -1.77 & -0.17 & 3.12 \end{bmatrix}$$

$$\begin{aligned} H^{-T} l_1 &= H^{-T}(-1, 0, 0.05)^T \\ &= (-1.1, -0.39, 1.92)^T \\ &= k(-0.94, -0.34, 1.66)^T \\ &= l'_1 \end{aligned}$$

$$\begin{aligned} H^{-T} l_2 &= H^{-T}(0, -1, 0.05)^T \\ &= (0.14, -0.21, 0.33)^T \\ &= k(0.54, -0.84, 1.35)^T \\ &= l'_2 \end{aligned}$$

$$\begin{aligned} H x_3 &= H^{(}0.05, 0.05, 1)^T \\ &= (1.11, 2.55, 1.15)^T \\ &= k(0.97, 2, 22, 1)^T \\ &= x'_3 \end{aligned}$$

Vanishing point, revisited



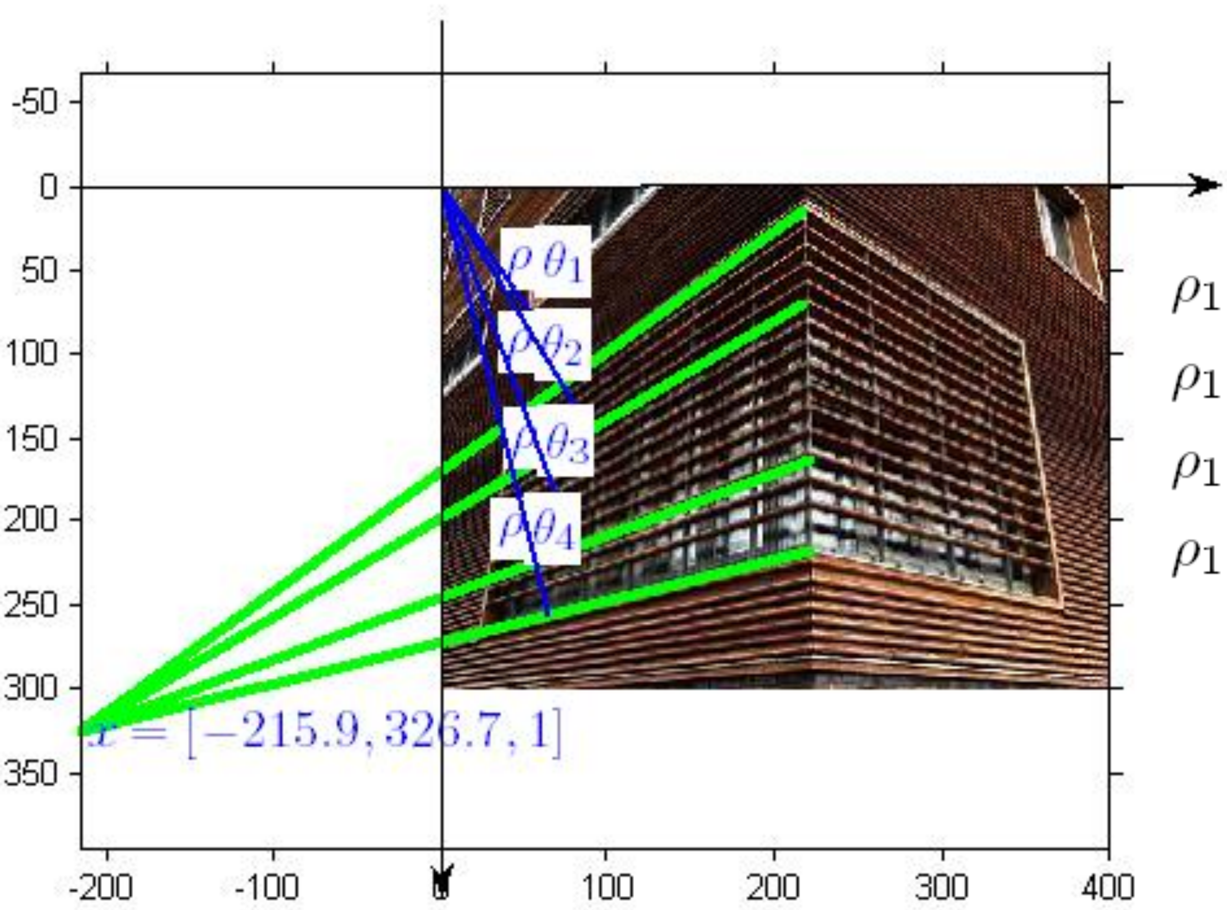
Points at infinity: Revisited

- Where are the points at infinity in the image plane?
  - The point at infinity can be in the ***FINITE*** region of the image !

$$\begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} \Rightarrow x'_3 \neq 0$$

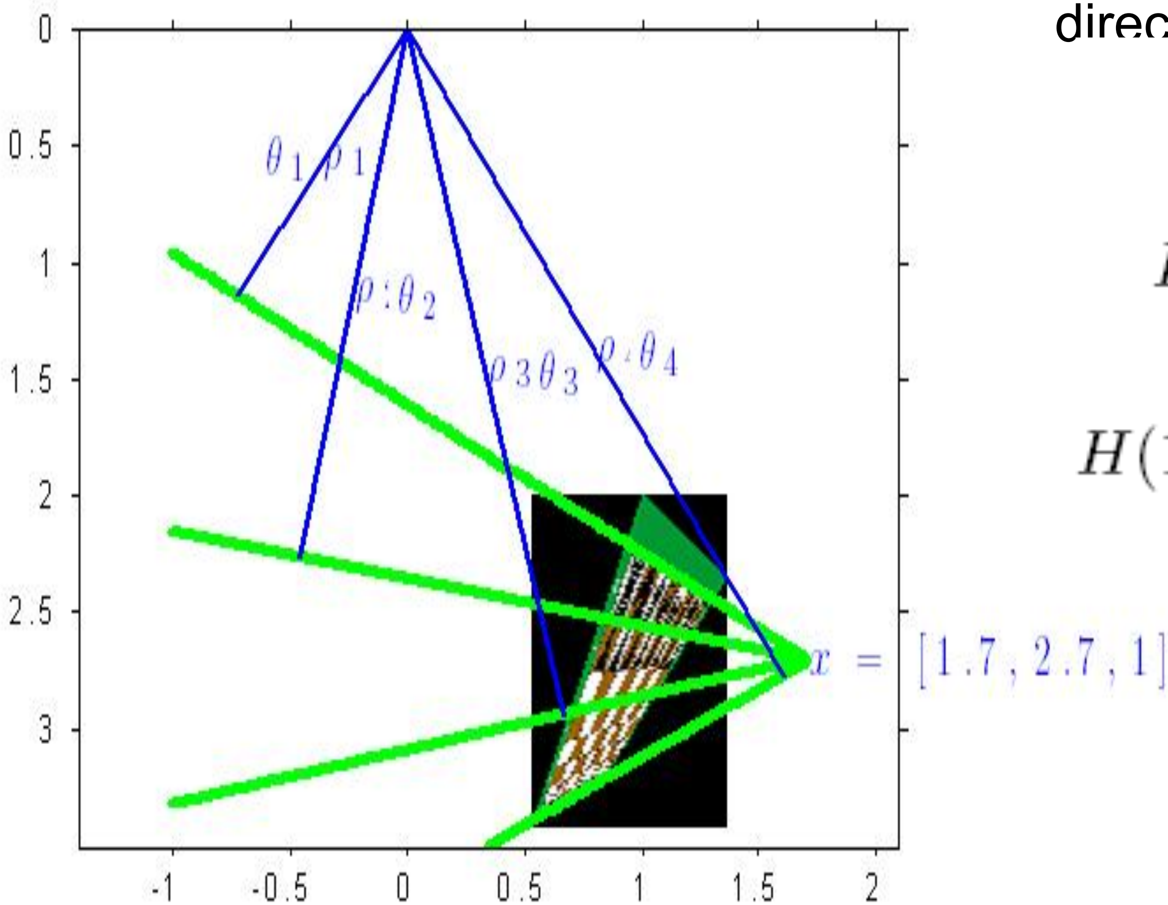
$$\text{if } h_{31}, h_{32} \neq 0$$

Example



$$\rho_1 = 139.0 \quad , \quad \theta_1 = 0.30\pi$$
$$\rho_1 = 172.3 \quad , \quad \theta_1 = 0.33\pi$$
$$\rho_1 = 228.6 \quad , \quad \theta_1 = 0.39\pi$$
$$\rho_1 = 264.9 \quad , \quad \theta_1 = 0.42\pi$$

Seeing vanishing point



- vanishing point of horizontal direction:

$$(b, -a, 0)^T = (1, 0, 0)$$

$$H = \begin{bmatrix} 1.71 & 0.586 & 1 \\ 2.71 & 8.24 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$H(1,0,0)^T = (1.7, 2.7, 1)^T$$

$$\rho_1 = 1.35 \quad , \quad \theta_1 = 0.68\pi$$

$$\rho_1 = 2.31 \quad , \quad \theta_1 = 0.56\pi$$

$$\rho_1 = 3.01 \quad , \quad \theta_1 = 0.43\pi$$

$$\rho_1 = 3.20 \quad , \quad \theta_1 = 0.33\pi$$

Line at infinity: Revisited

- A line passing all points at infinity:

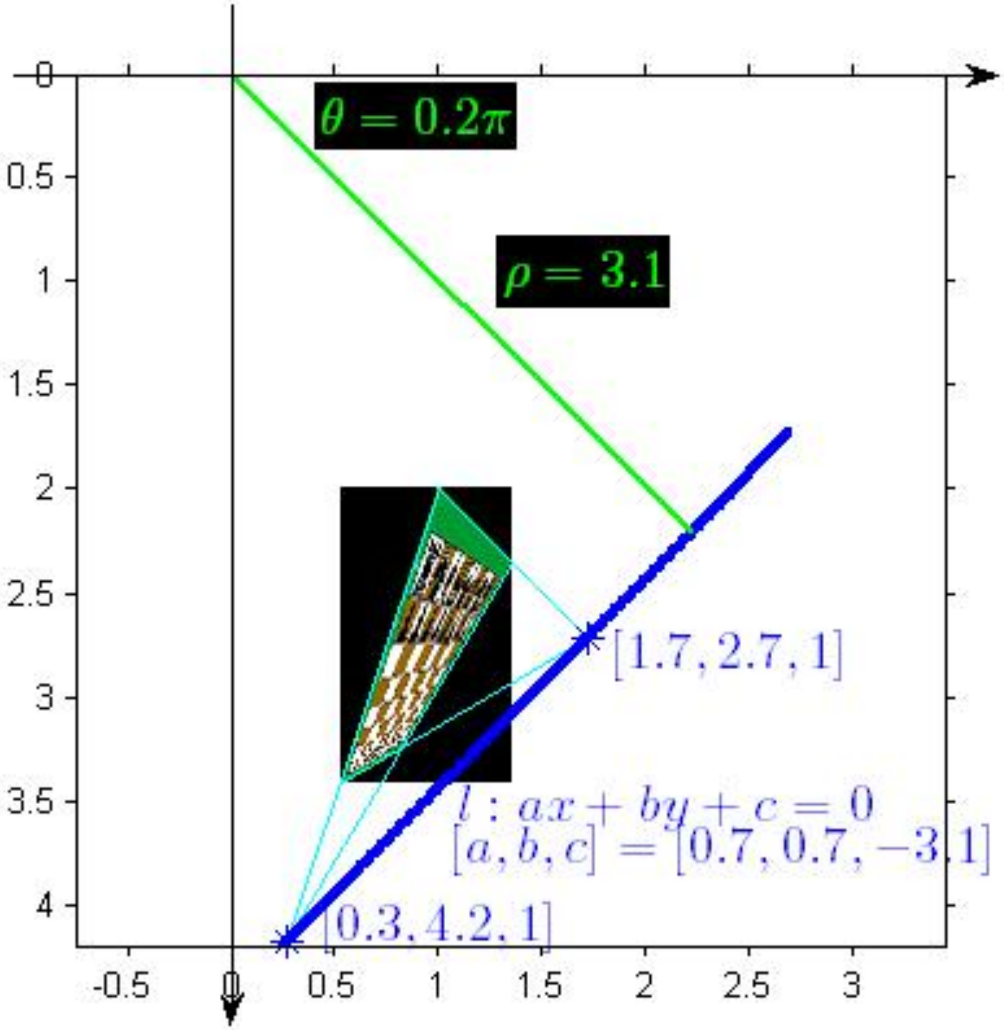
$$l_\infty = (0, 0, 1)^T$$

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = 0$$

- In the image plane:

$$l'_\infty = H^{-T} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Line of infinity

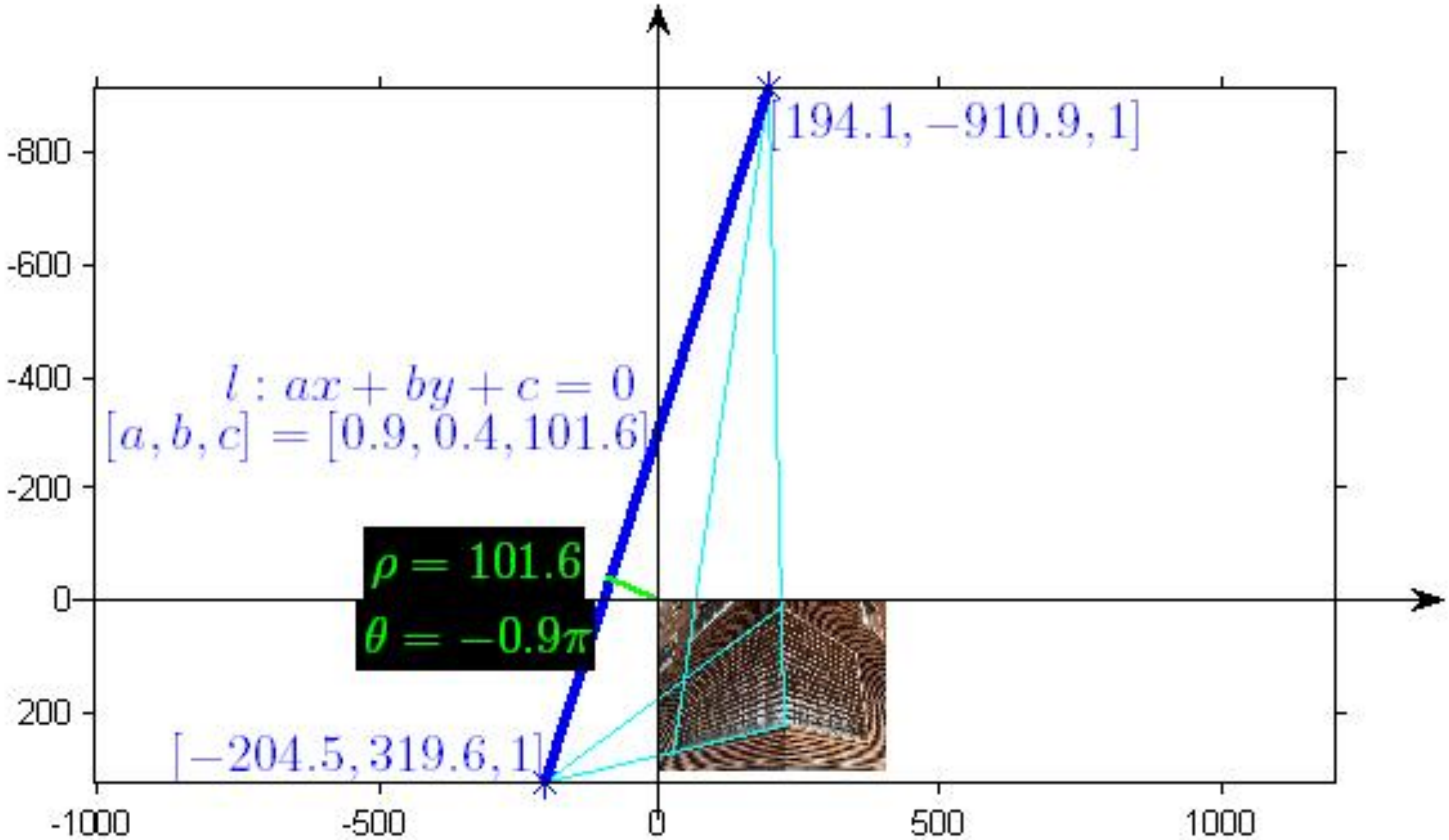


- line of infinity is:

$$\begin{aligned} l_\infty &= (0, 0, 1)^T \\ H^{-T} &= \begin{bmatrix} 1.06 & -0.18 & -0.71 \\ 0.35 & 0.18 & -0.71 \\ -1.77 & -0.17 & 3.12 \end{bmatrix} \\ H^{-T} l_\infty &= -1(0.7, 0.7, 3.1)^T \end{aligned}$$

Notice: this is the last column of H^{-T}

The line of infinity



- Extra slides